

‘Third’ Quantization of Vacuum Einstein Gravity and Free Yang–Mills Theories¹

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Received June 28, 2006; accepted August 30, 2006
Published Online: February 22 2007

Certain pivotal results from various applications of Abstract Differential Geometry (ADG) to gravity and gauge theories are presently collected and used to argue that we already possess a geometrically (pre)quantized, second quantized and manifestly background spacetime manifold independent vacuum Einstein gravitational field dynamics. The arguments carry also *mutatis mutandis* to the case of free Yang–Mills theories, since from the ADG-theoretic perspective gravity is regarded as another gauge field theory. The powerful algebraico-categorical, sheaf cohomological conceptual and technical machinery of ADG is then employed, based on the fundamental ADG-theoretic conception of a field as a pair $(\mathcal{E}, \mathcal{D})$ consisting of a vector sheaf \mathcal{E} and an algebraic connection \mathcal{D} acting categorically as a sheaf morphism on \mathcal{E} 's local sections, to introduce a ‘universal’, because expressly functorial, field quantization scenario coined *third quantization*. Although third quantization is fully covariant, on intuitive and heuristic grounds alone it formally appears to follow a canonical route; albeit, in a purely algebraic and, in contradistinction to geometric (pre)quantization and (canonical) second quantization, manifestly background geometrical spacetime manifold independent fashion, as befits ADG. All in all, from the ADG-theoretic vantage, vacuum Einstein gravity and free Yang–Mills theories are regarded as external spacetime manifold unconstrained, third quantized, pure gauge field theories. The paper abounds with philosophical smatterings and speculative remarks about the potential import and significance of our results to current and future Quantum Gravity research. A postscript gives a brief account of this author’s personal encounters with Rafael Sorkin and his work.

KEY WORDS: quantum gravity; quantum gauge theories; geometric (pre) quantization; second quantization; abstract differential geometry; sheaf theory; sheaf cohomology; category theory.

PACS Numbers: 04.60.-m, 04.20.Gz, 04.20.-q.

¹ Wholeheartedly dedicated to Rafael Sorkin on the occasion of his 60th birthday. Posted at the e-archives: gr-qc/0606021.

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1. MOTIVATIONAL REMARKS

Modern fundamental physics may be cumulatively referred to as ‘*field physics*’. The theoretical concept of ‘*field*’ is the cornerstone of our most successful and experimentally verified theories of Nature: from the macroscopic General Relativity (GR) describing gravity which shapes the large scale structure of the Universe, to the microscopic Quantum Field Theory (QFT) describing the structure and dynamical transmutations of matter at subatomic scales (Auyang, 1995; Cao, 1997).

At the same time, field theory in general, at least as it has been thought of and practiced almost ever since its inception until today, appears to be inextricably tied to a background manifold, which is physically interpreted as the ‘spacetime continuum’—be it for example the curved Lorentzian spacetime manifold of GR, or the flat Minkowski space of the flat (:gravity-free) QFTs of matter. Indeed, the current theoretical consensus maintains that it takes a mathematical continuum such as a locally Euclidean space⁴ to accommodate systems with an infinite number of degrees of freedom—the currently widely established conception of fields. The bottom line is that field theory, at least regarding the mathematical means that we have so far employed to formulate it, relies heavily on the notions, methods and technology of Classical Differential Geometry (CDG), which in turn is vitally dependent on (the *a priori* assumption of) a base differential manifold to support its concepts and constructions (Göckeler and Schücker, 1990; Kriele, 1999). Let us reduce this to a ‘boxed slogan’:

S1. The basic mathematical framework of field theory is the CDG of smooth manifolds.

In fact, such has been the influence of CDG on the development of field theory (and vice versa!) that it is not an exaggeration to say that *it is almost impossible to think of the latter apart from the former*. One should consider for instance the immense influence that the modern developments of CDG in terms of smooth fiber bundles have exercised on the way we view and treat (classical or quantum) gauge field theories of matter, including gravity (Ivanenko and Sardanashvily, 1983; Göckeler and Schücker, 1990; Auyang, 1995; Cao, 1997; Kriele, 1999).

Of course, that the theoretical physicist has so readily adopted the mathematics of CDG may be largely attributed to the fact that her principal aim—ideally, to discover and describe (:mathematically model) the laws of Nature—coupled to her theoretical requirement that the latter be *local* mathematical expressions, have found fertile ground in the manifold based CDG, as our second boxed slogan posits:

⁴Finite (e.g., spacetime) or infinite-dimensional manifolds (e.g., the fields’ configuration spaces).

S2. Physical laws are to be modelled after differential equations.

For example, more than a century ago, Bertrand Russell (1948) went as far as to maintain that

“*The laws of physics can only be expressed as differential equations.*”

Indeed, the background geometrical locally Euclidean continuum (be it spacetime, or the field’s configuration space) provides one with a smooth geometrical platform on which the apparently necessary *principle of infinitesimal locality*—the *a priori* theoretical requirement for a smooth causal nexus between the world-events triggered by contiguous field actions—can be snugly accommodated and (differential) geometrically pictured (i.e., represented by differential equations and their smooth solutions)

At the same time, few theoretical/mathematical physicists—general relativists and quantum field theorists alike—would disagree that the pointed background geometrical spacetime manifold is the main culprit for various pathologies that GR and QFT suffer from, such as singularities and related unphysical infinities (Clarke, 1993). In principle, any point of the underlying manifold can be the *locus* of a singularity of some physically important smooth field—a site where the field seems to blow up uncontrollably without bound and the law (:differential equation) that it obeys appears to break down somehow. Given that few physicists would actually admit such divergences (:infinities) as being physical, it is remarkable that even fewer would readily abandon the manifold based CDG as a theoretical/mathematical framework in which to formulate and work out field theories. They would rather resort to manifold and, *in extenso*, CDG-conservative effective approximation (e.g., perturbation) methods and would take great pains to devise quite sophisticated regularization and renormalization techniques to cope with the infinities, instead of doing away once and for all with the background geometrical spacetime manifold M . In view of S1, this is understandable, because if M will have to go, so will field theory, and then how, other than by differential equations proper set up by CDG-means, would physical laws be represented (S2)? *Mutatis mutandis* then for the geometrical picturization of the local field and particle dynamics: how, other than the usual imagery depicting the propagation and interaction of fields (and their particles) on a continuous base spacetime arena, could one geometrically picture (:represent) the field and particle dynamics?⁵

⁵Think for example of the physicist’s ‘archetypical’ theoretical image of *dynamical paths* (:trajectories) traversed by particles during their dynamical evolution. These are normally taken to be smooth curves in a 4-dimensional space-time continuum.

Especially when it comes to Quantum Gravity (QG), the aforesaid resort to CDG-conservative means may be justified on a reasonable analogy (A) and its associated hopeful expectation (E); namely that,

- **A.** Much in the same way that the background manifold conservative quantization of the classical CDG-based field theories of matter managed to alleviate or even remove completely the unphysical infinities via ‘analytic’ renormalization (e.g., QED ‘resolving’ the infinities of Maxwellian electrodynamics), so QG—regarded as Quantum General Relativity (QGR)⁶—could (or more demandingly, should!) remove singularities and their associated infinities.
- **E.** Thus, all we have to aim and hope is for a better, more subtle, refined and powerful ‘*Analysis*’⁷—perhaps one with formal quantum traits inherent in its formalism; albeit, one that still essentially relies on a background geometrical manifold in one way or another,⁸ for *how else could one do field theory*—be it *quantum field theory—differential geometrically* (S1)?

Alas, QG has proven to be (perturbatively) non-renormalizable, plus it appears to mandate the existence of a fundamental space-time length-duration—the Planck scale—below which the spacetime continuum is expected to give way to something more reticular and quantal: ‘*quantized spacetime*’, so to speak. So, *there goes our cherished field-theoretic, CDG-based, outlook on QG?* Not quite, yet.

Prima facie, in view of the existence of Planck’s fundamental cut-off space-time scale, that QG (viewed and treated as QGR) is non-renormalizable is not a blemish after all. Indeed, the (perturbatively) *renormalizable* (flat) continuum based (quantum) gauge theories describing the other three fundamental forces do *not* have such dimensionful constants (:space-time scales, or ‘coupling’ constants combining to produce those scales) inherent in their theoretical fabric. In turn, the Planck length is the *raison d’être et de faire* of non-perturbative QGR. The latter, at least in its present and most promising gauge-theoretic formulation as Loop Quantum Gravity (LQG) and Cosmology (LQC) (Rovelli and Smolin, 1990; Thiemann, 2001, 2002; Smolin, 2004) which is based on the Ashtekar formalism for GR (1986), effectively removes the continuous manifold picture of space-time and resolves the singularities-*cum*-infinities that the latter is responsible for

⁶ Here, by QGR we understand in general QG approached as a QFT—a quantum (or quantized, canonically and/or covariantly) field theory of gravity on a background differential spacetime manifold (Thiemann, 2001, 2002), with the classical theory being GR (on the same background!; see below).

⁷ Hereafter, the terms ‘CDG’, ‘Analysis’ and ‘Differential Calculus on Manifolds’ shall be regarded as synonyms and used interchangeably.

⁸ Here we have in mind various attempts at applying a ‘quantized’ (e.g., ‘noncommutative’) sort of Calculus to quantum spacetime, gravity and gauge theories—Connes’ Noncommutative (Differential) Geometry being the ‘canonical’ example of such an enterprise (Kastler, 1986; Connes, 1994, 1998; Chamseddine and Connes, 1996, 1997; Madore, 1999).

by means of ‘*spacetime quantization*’ (Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997a,b; Thiemann, 1998; Bojowald, 2001; Husain and Winkler, 2004; Modesto, 2004). However, the mathematical formalism devised recently to formalize and carry out that quantization, Quantum Riemannian Geometry (QRG) (Ashtekar, 2003), is still drawing amply from a background differential manifold for its differential geometric expression. All in all, the QRG-based LQG and LQC fulfill the aforementioned expectations (A,E), and what’s more, without resorting to perturbative renormalization arguments, techniques or results. *En passant*, let it be noted here that the other approach to (non-perturbative) QG currently competing with LQG for popularity (and monarchic hegemony!), (super)string theory (perturbative or not), also heavily relies on the manifold based CDG for its concepts and techniques. One should think for example of how higher-dimensional (real analytic or holomorphic) differential manifolds such as Riemann surfaces, Kähler spaces, Calabi-Yau manifolds, \mathbb{Z}_2 -graded manifolds (:supermanifolds), etc., have become the bread and butter mathematical structures in current string and brane theory research.

With the remarks above in mind, an overarching theoretical requirement or ‘principle’ underlying most (if not all) of the current (non-perturbative) QG approaches, including LQG and string theory, is that of *background independence* (Ashtekar and Lewandowski, 2004; Smolin, 2005; Seiberg, 2006). Expressed as a theoretical imperative:

B. The true quantum gravity must be a background independent theory.

The original requirement for ‘background independence’ pertained to ‘background *geometry* independence’—i.e., that QG should be formulated in a background *metric* independent way. Lately, the term ‘geometry’ is understood and used in the broader (mathematical) sense of (a structureless set endowed with some) ‘structure’ (MacLane, 1986), so that a background independent formulation of QG means one that does not employ any fixed background structure—an ‘absolute geometrical space’ of any kind.⁹ These two conceptions of background independence—the old, stricter and ‘weaker’ one, and the new, generalized and

⁹ Hereafter, let us call a set equipped with some structure a (mathematical) ‘*space*’. This is pretty much how a ‘geometrical space’ has been conceived in the physics (Stachel, 2003) and mathematics (MacLane, 1986) literature. Abiding by set-theoretic notions is not necessary, however. For example, the novel ‘quantization on a category’ scheme recently proposed by Isham (2003, 2004a,b, 2005), may still be perceived as being background dependent in the strict sense of B2 (see the next paragraph in the main text); albeit, the background is not a point-set proper, but a category—a mathematical universe of generalized (and in a certain sense, variable!) sets, as well as of maps (:morphisms) between them. A die-hard background independent ‘quantum gravitist’ might still regard such a scheme as being background dependent in disguise. But let us set aside such ‘extremist’ and ‘purist’ views, and plough on. For in any case, the backgrounds involved in Isham’s work (e.g., discrete topological spaces or causal sets) are far from being smooth manifolds, and are by no means fixed.

‘stronger’ one—are pretty much how they have been recently expressed in Kribbs and Markopoulou (2005) as a distillation from Butterfield and Isham (2000):

B1. ‘Weak’ Background Independence (WBI): “...Background independence

1. A quantum theory of gravity is background independent if its basic quantities and concepts do not presuppose a background metric.”

B2. ‘Strong’ Background independence (SBI): “Background independence

2. A quantum theory of gravity is background independent if there is no fixed theoretical structure. Any fixed structure will be regarded as a background...”

There are strong Leibnizian undertones in SBI, in the following sense: the true QG must be formulated in a *relational* way, without reference or recourse to any ‘absolute’, ether-like background structure (e.g., ‘spacetime’) whatsoever (Smolin, 2005).

In this respect, it is fair to say that so far (non-perturbative) string theory has *not* managed to achieve a background independent formulation of QG even in the restricted (B1) sense, since the whole formalism and interpretation of the theory vitally depends on a background (usually taken to be Minkowski) metric (space). Even LQG, although it is background metric independent (WBI), it is not (yet) background independent in the stronger sense (SBI), since, as noted earlier, its formulation relies heavily on manifold based CDG-means—i.e., the background in this case being the geometrical differential (spacetime) manifold and thus the theoretical/mathematical framework employed is effectively the CDG of such smooth ‘domains’. The spacetime continuum and its pathologies (e.g., singularities and associated infinities) is indeed evaded, but, as mentioned briefly before, only after a canonical-type of quantization procedure is exercised on the classical theory (:GR) and its supporting spacetime continuum (Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997a,b; Thiemann, 1998; Bojowald, 2001; Modesto, 2004; Husain and Winkler, 2004).

On the other hand, we have *Abstract Differential Geometry* (ADG)—the purely algebraico-categorical (:sheaf-theoretic) framework in which one can do differential geometry in a manifestly background manifold independent, thus effectively Calculus-free, way (Mallios, 1998a,b, 2005). Indeed, in a Leibnizian–Machean sense (Mallios and Raptis, 2005), the entire differential geometric ADG-machinery focuses on, and derives directly from, the algebraically (i.e., sheaf theoretically) represented *dynamical* relations between the ‘*geometrical objects*’ that ‘live’ on ‘space(time)’—the dynamical fields themselves—without that background ‘space(time)’ playing *any* role, thus having no physical significance whatsoever, in the said field dynamics (Mallios, 1998a,b, 2005; Mallios and Raptis, 2001, 2002, 2003, 2005). Moreover, the dynamics is still represented by differential equations proper between the ADG-fields; albeit, the latter are abstract,

algebraico-categorical expressions involving equations between sheaf morphisms that the fields are modelled after, without recourse to a base spacetime manifold arena for their geometrical support and interpretation (:‘spacetime picturization’). So far, ADG has enjoyed numerous applications to gauge (:Yang–Mills) theories and gravity (Mallios, 1998a,b, 2001, 2002, 2005; Mallios and Raptis, 2001, 2002, 2003, 2005; Raptis, 2006a,b,c).

In the present paper we employ the purely algebraic (:sheaf-theoretic) and manifestly background (spacetime) manifoldless concepts and technology of ADG in order to arrive at a ‘universal’, because manifestly functorial, field-quantization scenario for (free) Yang–Mills fields, including (vacuum) Einstein gravity which from an ADG-theoretic perspective is regarded as another gauge theory (Mallios, 2001, 2005; Mallios and Raptis, 2001, 2002, 2003, 2005; Raptis, 2006a,b,c). The basic sheaf-theoretic machinery is *sheaf cohomology*, while the scenario formally resembles canonical quantization, but it expressly avoids any mention of or reference to a background geometrical (spacetime) manifold structure. Rather, it is an entirely relational (:algebraic) scheme since, as befits ADG, it concerns solely the ADG-fields (vacuum gravitational and free Yang–Mills) involved. In particular, the said canonical quantization-type of scenario involves our positing non-trivial local commutation relations between certain characteristic local (:differential) forms that uniquely characterize sheaf cohomologically the ADG-gauge fields and their particle-quanta. In turn, in a heuristic way these forms may be physically interpreted as abstract position and momentum determinations (:‘observables’), hence the epithet ‘canonical’ adjoined to the noun ‘quantization’ above. The base spacetime manifoldless sheaf cohomological ADG-field quantization proposed here is coined ‘*third quantization*’ so as to distinguish it from the usual manifold and CDG-based 2nd and, of course, 1st-quantization.

The paper is organized as follows: in the next section (2) we recall certain pivotal results from various applications of ADG to vacuum Einstein gravity (VEG) and free Yang–Mills (FYM) theories; in particular, to the geometric (pre)quantization and second quantization thereof. Based on these results, we then maintain that we already possess a geometrically (pre)quantized and second quantized vacuum Einstein gravitational and free Yang–Mills field dynamics. Especially, we highlight how the background spacetime manifold independent ADG-formalism enables us to:

1. Extend the current so-called ‘gauge theory of the second kind’ (:local gauge field theory) to what is here coined ‘*gauge field theory of the 3rd kind*’ (Mallios and Raptis, 2005; Raptis, 2006a,b,c), which, although manifestly local like its predecessor, it is not local(ized) on an external (to the gauge fields themselves) geometrical spacetime continuum.
2. Effectively *halve the order of the formalism*, since in our scheme the purely gauge connection field \mathcal{D} , and not the metric field $g_{\mu\nu}$ (or equivalently, the

tetrad field e_μ), is the sole dynamical variable (Mallios and Raptis, 2003, 2005; Raptis, 2006a,b,c). This enables us to contrast our purely gauge-theoretic ADG-gravitational formalism against the *manifestly background differential spacetime manifold dependent*, hence also CDG-based, second (:Einstein) and first order (Ashtekar-Palatini) formalisms. Fittingly, we coin our approach ‘ $\frac{1}{2}$ -order formalism’.

3. Pave the way towards *3rd-quantization*, by evading altogether a background spacetime manifold (thus also the CDG-based 1st and 2nd-quantization scenaria), and by concentrating instead on local algebraic dynamical relations between the ADG-fields involved ‘in-themselves’.¹⁰

Thus, in Section 3 we entertain the possibility of extending 2nd to 3rd-quantization in a way that suits the formal ADG-gauge field theory of the 3rd kind and its physical semantics. Third quantization is a heuristic conception of a dynamically autonomous, because external (:background) spacetime manifoldless and thereby unconstrained, ADG-gauge field quantization scenario which is tailor-cut for the geometrically (pre)quantized and second quantized ADG-field semantics. It formally appears to follow a canonical route, since it involves non-trivial local commutation relations between certain characteristic local (:differential) forms that uniquely characterize sheaf cohomologically the ADG-gauge fields and, from an ADG perspective on 2nd and geometric (pre)quantization, their particle-quanta. In turn, as noted above, in a heuristic way the said forms may be physically interpreted as abstract position and momentum determinations, hence the commutation relations that we impose on them may be regarded as *abstract sheaf cohomological Heisenberg uncertainty relations*. Of course, the formal analogy with the 2nd canonical field-quantization of gravity and gauge field theories stops here since, in glaring contrast to those two scenaria, our scheme is manifestly background spacetime manifold-free and it thus regards gravity as a pure (i.e., external spacetime continuum unconstrained) quantum gauge theory. In this way, gauge theory of the 3rd kind and 3rd-quantization appear to go hand in hand.

We also emphasize the manifest *functoriality* of 3rd-quantization, and then we draw preliminary, albeit suggestive, links between it and Mallios’ ADG-based *K*-theoretic treatment of geometric (pre)quantization and second quantization in Mallios (2004, 2005). Based on these *K*-theoretic smatterings, we highlight close affinities between 3rd-quantized VEG and FYM, and our ADG-based *finitary, causal and quantal VEG and FYM* in Mallios and Raptis (2001, 2002, 2003,

¹⁰ This ‘*autonomy*’ of 3rd-quantization is its essential feature, and it makes us think of the ADG-fields as quantum dynamically autonomous (:self-supporting; physical laws’ self-legislating) entities reminiscent of Leibniz’s ‘entelechian monads’ (Leibniz, 1992). This analogy is consistent with the Leibnizian (:purely relational, i.e., algebraic) conception of the differential geometry (:Differential Calculus) that ADG propounds (cf. Mallios and Raptis, 2005; Mallios, 2002, 2006a,b for an extensive discussion on this). The autonomy of the dynamical ADG-fields will be further corroborated by our 3rd-quantization scenario in the sequel.

2005) and Raptis (2006a,b,c). In the concluding Section 4, we summarize our findings and discuss briefly the potential impact that 3rd-quantization may have on certain outstanding (and persistently resisting resolution!) problems in current and future QG research.

2. GEOMETRICALLY (PRE)QUANTIZED AND SECOND QUANTIZED VACUUM EINSTEIN GRAVITY AND FREE YANG–MILLS THEORIES

From the ADG-theoretic perspective, vacuum Einstein gravity (VEG) is regarded and treated as a pure gauge theory, like the free Yang–Mills theories (FYM) (Mallios, 1998a,b, 2001; Mallios and Raptis, 2001, 2002, 2003, 2005; Raptis, 2006a,b,c). This means that the sole dynamical variable in the theory is an algebraic \mathbf{A} -connection \mathcal{D} on a vector sheaf \mathcal{E} , the (Ricci scalar) curvature of which, $\mathcal{R}(\mathcal{D})$, obeys the equation

$$\mathcal{R}(\mathcal{E}) = 0 \tag{1}$$

The corresponding formalism has been coined '*half-order formalism*' and it should be contrasted against Einstein's original 2nd-order one, where the only gravitational dynamical variable is the smooth spacetime metric $g_{\mu\nu}$. Perhaps more importantly *vis-à-vis* current QG trends, ADG-gravity should be contrasted against the more recent Ashtekar-Palatini 1st-order formalism (Ashtekar, 1986), in which although the smooth connection assumes a more assertive and physically significant role, thus pronouncing more the gauge-theoretic character of gravity, the metric is still present in the guise of the smooth tetrad field e_μ .

In ADG-gravity, the Eq. (1) is obtained from varying with respect to \mathcal{D} an Einstein-Hilbert action functional $\mathcal{E}\mathcal{H}$ on the affine space $A_{\mathbf{A}}(\mathcal{E})$ of \mathbf{A} -connections, which may be formally identified with the configuration space in the theory Mallios and Raptis (2003). This is in contrast to both the 2nd and the 1st-order formalism in which variation of the Einstein-Hilbert functional with respect to the metric and, what amounts to the same, with respect to the *vierbein* field respectively, produces (1). Moreover, in the 1st-order formalism, variation with the connection field produces the auxiliary metric compatibility condition for the connection

$$\mathcal{D}g = 0 \tag{2}$$

By contrast, in ADG-gravity the \mathbf{A} -metric¹¹ is a physically secondary (i.e., a dynamically not primary) structure, while its compatibility with the \mathbf{A} -connection \mathcal{D} is optional to the theorist, and certainly *not* deducible variationally from the dynamical action, which is a functional of the connection exclusively.¹²

¹¹ Symbolized by ρ in the theory (Mallios, 1998a,b, 2001; Mallios and Raptis, 2003).

¹² To be sure, in ADG-gravity the usual '*metric compatibility of the connection*' condition (2) above is still observed, but in the other way round. That is to say, *if* the theorist chooses to impose

With Mallios (1998a,b, 2005) and Mallios and Raptis (2003) as reference guides to the technical symbols, terms, their definitions and associated construction details, direct comparison between (1) and the FYM equation

$$\delta\mathcal{F}(\mathcal{D}) = 0 \text{ or } \Delta\mathcal{F}(\mathcal{D}) = 0^{13} \tag{3}$$

as well as between the Einstein–Hilbert action functional $\mathfrak{E}\mathfrak{H}$ and the Y–M one $\mathfrak{Y}\mathfrak{M}$ on the corresponding affine spaces $A_{\mathbf{A}}(\mathcal{E})$ of \mathbf{A} -connections on the respective \mathcal{E} s involved, shows what was mentioned in the beginning, namely, that

from the ADG-theoretic vantage, VEG is a ‘pure’ gauge theory, like the FYM theory.

A more glaring contrast between ADG-gravity and the usual 1st and 2nd-order formulation of GR (both of which rely mathematically on the CDG of \mathcal{C}^∞ -smooth pseudo-Riemannian manifolds) is that it does not employ at all any background geometrical locally Euclidean space (:differential manifold) to formulate the theory differentially. Rather, it relies solely on purely algebraic-categorical (:sheaf-theoretic) means to formulate its concepts and develop its constructions. It follows that

the theory’s physical semantics does not involve any background spacetime continuum interpretation and its associated ‘geometrical picturization’ either.

All that is involved in ADG-gravity—its fundamental, ‘ur’ element so to speak—is the ADG-gravitational field \mathfrak{F} , which is defined as a pair

$$\mathfrak{F} := (\mathcal{E}, \mathcal{D}) \tag{4}$$

consisting of a *vector sheaf*—by definition, a locally free \mathbf{A} -module of finite rank n on an in principle arbitrary topological space X ,¹⁴ and a linear, Leibnizian *sheaf*

an \mathbf{A} -metric structure ρ in the theory (:on \mathcal{E}), then she might like to make sure that this metric is compatible with (i.e., it respects) the \mathbf{A} -connection \mathcal{D} , which is the primary dynamical structure on \mathcal{E} . Thus, in ADG-gravity one talks about the ‘*connection compatibility of the metric*’, which is equivalent to the following ‘*horizontality condition*’ for the connection on the tensor product vector sheaf $\mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}^*) = (\mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^*)^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}^*$ induced by the \mathbf{A} -connection \mathcal{D} on \mathcal{E} :

$\mathcal{D}\mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}^*)(\tilde{\rho}) = 0$, where $\tilde{\rho}$ effectuates the canonical \mathbf{A} -isomorphism $\mathcal{E} \xrightarrow{\tilde{\rho}} \mathcal{E}^*$ between \mathcal{E} and its dual $\mathcal{E}^* \equiv \mathcal{H}om(\mathcal{E}, \mathbf{A}) \simeq \Omega$, Mallios and Raptis (2003).

¹³ Where $\mathcal{F}(\mathcal{D})$ is the curvature (:field strength) of the Yang–Mills connection \mathcal{D} , while δ and Δ the ADG-versions of the usual coderivative and Laplacian differential operators, respectively.

¹⁴ \mathbf{A} is a sheaf of unital, and associative *differential* (and not necessarily functional, strictly speaking) \mathbf{K} -algebras ($\mathbf{K} = \mathbb{R}_X, \mathbb{C}_X$: the constant sheaf of real or complex numbers over X) called the *structure sheaf of generalized arithmetics* (the terms ‘coefficients’ and ‘coordinates’ are synonyms to ‘arithmetics’). By definition, \mathcal{E} is locally a finite power of \mathbf{A} : $\Gamma(U, \mathcal{E}) := \mathcal{E}(U) \equiv \mathcal{E}|_U \simeq \mathbf{A}(U)^n = \mathbf{A}^n(U)$, U open in X . At the same time, X is usually taken to be .

morphism \mathcal{D} , the \mathbf{A} -connection, acting on \mathcal{E} 's local sections in $\mathcal{E}(U)$. This is a particular instance of the general ADG-theoretic notion of a field \mathfrak{F} as a pair $(\mathcal{E}, \mathcal{D})$, which has been abstracted from the usual conception of a field as a connection on a smooth principal fiber (or its associated/representation vector) bundle.¹⁵

Due to the manifest absence in ADG-VEG of a smooth background spacetime manifold,

the ADG-gravitational field can be regarded as an external smooth spacetime unconstrained gauge system.

This is another result supporting our claim that ADG-gravity is a pure gauge theory. Moreover, this fact has profound consequences for plausible quantization scenaria within the ADG-framework as we shall argue in the sequel. For one thing, while the usual notions of 'space' and 'time' are *not* primary in ADG-field theory, they may still be thought of as being 'inherent' in the *ur*-concept of ADG-field. For example, 'space' may be thought of as being already effectively encoded in \mathbf{A} (e.g., Gel'fand duality) (Mallios, 2002, 2004, 2006a,b; Mallios and Raptis, 2003, 2005),¹⁶ while an abstract notion of 'time' (: 'dynamical change' or 'progression') is arguably already inherent in the dynamical evolution equation (1) for \mathcal{E} 's states (:local sections) on which the ADG-gravitational \mathbf{A} -connection field \mathcal{D} acts, via its (Ricci) curvature, to change them dynamically.

In fact, as it must have already been transparent from the exposition so far, one can adopt and adapt the entire gauge field-theoretic conceptual jargon and technical machinery to ADG-field theory, briefly as follows: one can cover the base topological space X by a system \mathcal{U} of *local open gauges* U and relative to it consider *local gauge frames* e^U ($U \in X$ open) constituting local bases of $\mathcal{E}(U)$.¹⁷ Then, in view of the aforementioned local isomorphism $\mathcal{E}(U) \simeq \mathbf{A}^n(U)$, one identifies the natural gauge (:structure) group sheaf (:principal sheaf Vassiliou, 1994, 1999, 2000; Mallios, 1998a, 2005) of the ADG-gauge field pair $(\mathcal{E}, \mathcal{D})$ with

$$\text{Aut}\mathcal{E}(U) = \text{End}\mathcal{E}(U)^\bullet = M_n(\mathbf{A}(U))^\bullet = \mathcal{GL}(n, \mathbf{A})(U) \quad (5)$$

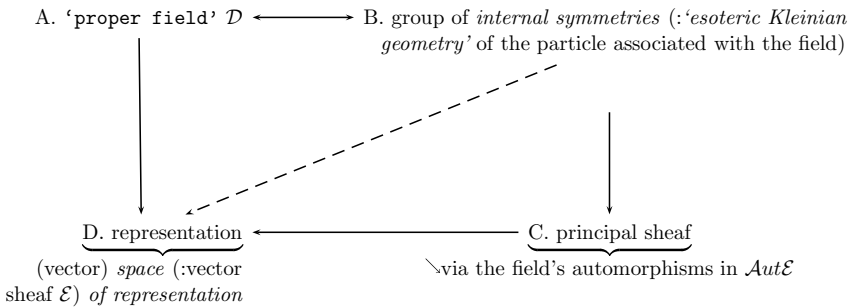
the group sheaf of local automorphisms of \mathcal{E} . This latter group sheaf effectuates in ADG-gravity the abstract version of the Principle of General Covariance (PGC), since it is the ADG-analogue of the Lie group $GL(4, \mathbb{R})$ of general coordinate

¹⁵ For example, the classical electromagnetic field of Maxwellian electrodynamics is regarded as the pair $(\mathcal{L}, \mathcal{D})$ consisting of a connection \mathcal{D} on a line bundle \mathcal{L} (:the associated bundle of the $U(1)$ principal fiber bundle of electrodynamics) (Manin, 1988). Analogously, the Maxwell field in ADG is defined as a connection on a line *sheaf* \mathcal{L} (:a vector sheaf of rank 1) (Mallios, 1998a,b, 2005; Mallios and Raptis, 2002, 2003).

¹⁶ Much like in the usual theory (:CDG), a differential manifold M can be derived from the structure sheaf $\mathbf{A} \equiv C_M^\infty$ of germs of smooth (\mathbb{R} -valued) functions on it as the latter's (real) Gel'fand spectrum.

¹⁷ That is, any local section $s \in \mathcal{E}(U)$ can be expanded as a unique linear combination of the n linearly independent local sections in e^U , with coefficients in \mathbf{A} .

transformations in the 4-dimensional spacetime manifold based GR (Mallios and Raptis, 2003, 2005; Raptis, 2006a,b,c). Moreover, the principal sheaf $\mathcal{Aut}\mathcal{E}$ is the sheaf-theoretic ADG-analogue of the ‘structure’ group $\text{Diff}(M)$ of the base differential spacetime manifold of GR,¹⁸ it too effectuating a ‘global’ version of the PGC of GR in ADG-gravity. In turn, as briefly mentioned before, \mathcal{E} may be regarded as the associated (*alias*, representation) sheaf of the principal sheaf $\mathcal{Aut}\mathcal{E}$, carrying a representation of the (local) structure group $\text{GL}(n, \mathbf{A})$ in its fibers (*alias*, stalks). We may summarize graphically the above in the following diagram, which we borrow from Mallios (2005):¹⁹



Note in the diagram above that by ‘proper field’ we refer to the connection part \mathcal{D} of the ADG-field pair $\mathfrak{F} = (\mathcal{E}, \mathcal{D})$. This separation and distinction between the ‘proper field’ (\mathcal{D}) and the ‘total field’ $\mathfrak{F} = (\mathcal{E}, \mathcal{D})$ may seem superfluous at first sight, but it is of great semantic significance: namely, in ADG the field proper is the connection \mathcal{D} existing ‘out there’ independently of us,²⁰ which is then geometrically materialized (:represented) by \mathcal{E} when we coordinatize it by introducing a structure sheaf \mathbf{A} of our choice. The deeper significance of this distinction will become clearer in the next section where we introduce our third ADG-field quantization scenario and we discuss the \mathbf{A} -functoriality of the ADG-gravitational dynamics (1), of 3rd-quantization in general, as well as the principle of ADG-field realism that this functoriality entails.

It is fitting to stress at this point that, in the past (Mallios and Raptis, 2003, 2005; Raptis, 2006a,b,c) the purely gauge field-theoretic ADG-perspective on

¹⁸ Indeed, by assuming C_X^∞ as structure sheaf \mathbf{A} in the theory, X can be identified with a smooth manifold M by Gel’fand duality as briefly noted earlier, and then plainly, by definition: $\text{Aut}M \equiv \text{Diff}(M)$.

¹⁹ I wish to thank Mrs Popi Mpoliotti, of the Algebra and Geometry Section, Maths Department, University of Athens, for giving me the LaTeX graphics for this ‘categorical/commutative’ diagram, and of course to Tasos Mallios for permitting me to borrow and slightly modify it from his latest book (Mallios, 2005).

²⁰ That is, independently of our generalized measurements (:‘coordinatizations’) in \mathbf{A} and associated geometrical representations by $\mathcal{E} \stackrel{\text{loc.}}{=} \mathbf{A}^n$.

gravity has been coined ‘*gauge theory of the third kind*’, due to the following features:

1. As noted earlier, the sole dynamical variable is the algebraic \mathbf{A} -connection \mathcal{D} ($:\frac{1}{2}$ -order formalism);
2. The scheme is manifestly local (:sheaf-theoretic) like the current gauge theories of the second kind, and in contradistinction to Weyl’s original gauge theory of the first kind, which was a global gauge (:scale) theory;
3. On the other hand, our ADG-gauge field theory is *not* local in the usual sense of the modern-day gauge theories of the second kind—i.e., the gauge transformations (and symmetries) are *not* localized over an external, base (:background) spacetime continuum (:manifold), since the latter does not exist in our theory. All there is in our scenario is the dynamically autonomous ADG-gravitational field $(\mathcal{E}, \mathcal{D})$, which does not depend on a background spacetime manifold, and solely in terms of which (and its curvature) the VEG dynamics is expressed as in (1);
4. It follows that in our scheme, unlike the physical semantics of nowadays gauge theories of the second kind, there is no distinction between external (‘spacetime’) and internal (‘gauge’) transformations (and dynamical symmetries). All our transformations (and dynamical symmetries) are ‘internal’ (:gauge), taking place within the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$, and are represented by $\mathcal{A}ut\mathcal{E}$.
5. The features above reveal an unprecedented *fundamental dynamical autonomy* in ADG-gravity, which is part and parcel of the theory’s genuine *background spacetime manifold independence*. Namely, all that ‘exists’ and is of physical significance in ADG-VEG (and FYM theories) is the autonomous dynamical field $(\mathcal{E}, \mathcal{D})$, the law that it obeys/defines as a differential equation proper (1)–(3), and the latter’s ‘dynamical self-invariances’ (:‘autosymmetries’) in $\mathcal{A}ut\mathcal{E}$, without any reference to or dependence on an extraneous (:externally imposed) structure (e.g., background spacetime manifold) to support that autodynamics.²¹

All the ideas above synergistically come to fruition when ADG is applied for the geometric (pre)quantization and second quantization of gauge theories, including gravity (Mallios, 1998a,b, 1999, 2004, 2005; Mallios and Raptis, 2002, 2003). Indeed, there \mathcal{E} is regarded as the Hilbert (or Fock) \mathbf{A} -module sheaf associated to the (principal) Klein group sheaf $\mathcal{A}ut\mathcal{E}$ of field automorphisms. The basic

²¹ Elsewhere, Mallios and Raptis (2003, 2005) and Raptis (2006a), we have coined this Leibnizian monad-type of dynamical autonomy of the ADG-field ‘dynamical holism’, or even, ‘unitarity’. Like Leibniz’s monads, the ADG-fields possess their own ‘dynamical entelechy’, but at the same time unlike them, they are not ‘windowless’, as they can (dynamically) interact with each other. Here, however, we have presented only the free field theories.

result there is the following identification:

$$\text{local quantum particle states of the field} \longleftrightarrow \text{local sections of } \mathcal{E}^{22} \quad (6)$$

which is then carried further to conclude that

every elementary field is geometrically (pre)quantizable and second quantizable, that is to say, it admits a prequantizing vector (Hilbert $\mathcal{E} \equiv \mathcal{H}$), or a second quantizing vector (Hilbert-Fock $\sum_{n \in \mathbb{Z}_+} \otimes_{\mathbf{A}}^n \mathcal{E}$), sheaf as the representation state space of its identical particle-quanta. In particular, the *spin-statistics connection* of the usual spacetime manifold and CDG-based QFTs of matter is observed in that local quantum particle states of boson (:integer spin) fields are represented by vector sheaves of rank $n = 1$ (:line sheaves), while those of fermion (:half-integer spin) fields by sections of vector sheaves of rank $n > 1$.

In view of these results, we can now claim that

(1) is a geometrically (pre)quantized and second quantized version of the VEG equations, for a suitable choice²³ of representation sheaf \mathcal{E} for the free gravitational quanta. The same holds for (3) and the free ('bare') gauge bosons carrying the other three fundamental gauge forces. On the other hand, matter quanta (e.g., electrons) have connections (e.g., the Dirac operator) defined on vector sheaves of rank greater than 1 (:Grassmannian \mathbf{A} -module sheaves).

In closing this section, we note as an *addendum* that when for example \mathbf{A} is taken to be the C^* -algebra sheaf of germs of continuous \mathbb{C} -valued functions on a compact Hausdorff topological space X (e.g. a compact C^0 -manifold), the Kleinian endomorphism algebra sheaf $M_n(\mathbf{A})(U)$ of the field may be regarded as the field's 'noncommutative geometry' à la Connes (1994) and Block (1998). The commutative coordinate functions in \mathbf{A} are promoted to 'noncommutative' ones, which now represent the field's intrinsic (dynamical) self-transmutations (:endo/automorphisms) in $M_n(\mathbf{A})(U) \equiv \text{End}\mathcal{E}|_U$. This observation will prove to be very important in the next section where we insist that the Heisenberg-type of canonical commutation relations defining third ADG-field quantization should close within $\text{End}\mathcal{E}$ —the noncommutative (dynamical) Kleinian 'auto-geometry' of the ADG-field 'in-itself'. The latter we may metaphorically call '*quantum field foam*' as it is the structural quality of the ADG-field that gives it its 'foamy', 'fuzzy', dynamically ever-fluctuating character.

²² Notice here the '*self-duality*' of the total ADG-field $\mathfrak{F} = (\mathcal{E}, \mathcal{D})$ in (5): \mathfrak{F} has a '*particle*' (: \mathcal{E}) and a '*proper field*' (: \mathcal{D}) aspect, and it has thus been referred to as a '*particle-field pair*' (Mallios, 2004, 2005; Mallios and Raptis, 2003, 2005). In turn, this self-duality of the total ADG-field pair \mathfrak{F} is an abstract version of the usual '*quantum-theoretic duality*' between the '*wave/momentum*' (here, \mathcal{D}) and '*particle/position*' (here, \mathcal{E}) aspects of quanta, as we shall argue in the next section in connection with our 3rd-quantization of ADG-fields. There, we shall see that ADG-fields are '*quantum self-dual*' or '*self-complementary*' entities (in the quantum sense of 'complementarity' due to Bohr)—another feature pointing to their (quantum dynamical) autonomy alluded to earlier.

²³ Effectively, for an \mathbf{A} chosen by the theorist, since \mathcal{E} is locally a finite power of \mathbf{A} .

3. THIRD QUANTIZATION OF GRAVITY AND YANG–MILLS THEORIES

One can carry the quantum physical interpretation of the ADG-gauge field pair $(\mathcal{E}, \mathcal{D})$ even further and envisage a canonical-type of field quantization scenario along sheaf cohomological lines in the following physically intuitive and mathematically heuristic way²⁴: we noted earlier that local sections of \mathcal{E} represent local quantum-particle states of the ADG-VEG field, while \mathcal{D} acts on them (via its curvature) to change them dynamically according to (1).

We can thus, continuing our anticipatory remarks in footnote 19, heuristically and intuitively interpret the local sections of \mathcal{E} as abstract *position* or *coordinate* determinations²⁵ of the (particle-quanta of the) field; while, as befits the (generalized) differential operator \mathcal{D} , interpret its effect/action on those position states as an abstract *momentum* map.²⁶

Now, since the ADG-fields $(\mathcal{E}, \mathcal{D})$ are dynamically self-supporting, autonomous entities as we emphasized earlier; moreover, since they are ‘self-dual’ as it was anticipated in footnote 19,

a possible quantization scenario for them should involve solely their two constitutive parts, namely, \mathcal{E} and \mathcal{D} , without recourse to/dependence on extraneous structures (e.g., a base spacetime manifold) for its (physical) interpretation.

Thus, in what formally looks like a canonical quantization-type of scenario,

we envisage abstract non-trivial local commutation relations between the abstract position ($:\mathcal{E}$) and momentum ($:\mathcal{D}$) aspects of the ADG-fields.

To this end, we recall that

²⁴The arguments to be presented below are conceptual, informal and tentative, and should await a more formal and mathematically rigorous exposition. We shall do this in a forthcoming paper (Raptis, 2006d) (see the declaration at the end).

²⁵After all, \mathcal{E} is locally of the form \mathbf{A}^n , and as noted earlier, \mathbf{A} represents our abstract (local) coordinatizations ($:\text{local coordinate determinations or ‘measurements’}$) of the proper ADG-field \mathcal{D} . In turn, \mathcal{E} is the carrier (*alias*, representation) space of the proper field \mathcal{D} .

²⁶After all, momentum is normally perceived as a (‘rate’ of) change of position. Moreover, it must be noted here parenthetically that since the topological base space X plays no important role in the differential geometric mechanism of the theory, but it merely serves as a scaffolding or ‘surrogate space’ for the sheaf-theoretic localization of the ADG-fields (for instance, since the gravitational dynamics (1) is expressed categorically as an equation between \mathbf{A} -sheaf morphisms such as the curvature of the connection, which is an $\otimes_{\mathbf{A}}$ -tensor, it ‘sees through’ X ; see remarks on \mathbf{A} -functoriality later in this section), there is *no* notion of *tangent space* to it in ADG. It follows that the local sections of \mathcal{E} should *not* be interpreted as *tangent vectors* like in the usual theory ($:\text{CDG}$) of vector bundles over a smooth base manifold (e.g., the tangent bundle TM); hence the theory does *not* accommodate *derivations*, which are normally defined as maps $\text{Der} : \mathbf{A} \rightarrow \mathbf{A}$ and are represented by tangent vectors to the continuum. The abstract momentum maps noted above are *not* derivations in the usual ($:\text{CDG}$, fiber bundle-theoretic) sense.

there are certain local (:differential) forms that uniquely characterize sheaf cohomologically the vector sheaf \mathcal{E} and the connection \mathcal{D} parts of the ADG-fields $(\mathcal{E}, \mathcal{D})$

Thus, the basic intuitive idea here is to identify the relevant forms and then posit non-trivial commutation relations between them. Moreover, for the sake of the aforementioned ‘*dynamical ADG-field autonomy*’, we would like to require that

the envisaged commutation relations should not only involve just the two components (i.e., \mathcal{E} and \mathcal{D}) of the total ADG-fields \mathfrak{F} , but they should also somehow *close within the \mathfrak{F} s themselves*—i.e., the result of their commutation relations should not take us ‘outside’ the total ADG-field structure (and its ‘auto-transmutations’), which anyway is the only dynamical structure involved in our theory.²⁷

Keeping the theoretical requirements above in mind, we recall from (Mallios, 1998a,b, 1999, 2005; Mallios and Raptis, 2002) two important sheaf cohomological results:

1. That, sheaf cohomologically, the vector sheaves \mathcal{E} are completely characterized by a so-called *coordinate 1-cocycle* $\phi_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A}))$ associated with any system \mathcal{U} of local gauges of \mathcal{E} . Intuitively, this can be interpreted in the following Kleinian way: since any (vector) sheaf is completely determined by its (local) sections,²⁸ one way of knowing the latter is to know how they transform—in passing, for example, from one local gauge $(U_\alpha \in \mathcal{U})$ to another $(U_\beta \in \mathcal{U})$, with $U_\alpha \cap U_\beta \neq \emptyset$ and \mathcal{U} a chosen system of local open gauges covering X .²⁹ To know something (e.g., a ‘space’) is to know how it transforms, the fundamental idea underlying Klein’s general conception of ‘geometry’. The bottom-line here is that the characteristic cohomology classes of vector sheaves \mathcal{E} are completely

²⁷ This loosely reminds one of the theoretical requirement for algebraic closure of the algebra of quantum observables in canonical QG, with the important difference however that the $\text{Diff}(M)$ group of the external (to the gravitational field) spacetime manifold must also be considered in the constraints, something that makes the desired closure of the observables’ algebra quite a hard problem to overcome (Thiemann, 2001). Later on, we shall return to discuss certain difficult problems that $\text{Diff}(M)$ creates in various QG approaches, as well as how its manifest absence in ADG-gravity can help us bypass them totally. For, recall that from the ADG-perspective gravity is an external (:background) spacetime manifold unconstrained pure gauge theory (:of the 3rd kind).

²⁸ A basic motto (:fact) in sheaf theory is that “*a sheaf is its sections*” (Mallios, 1998a). If we know the local data (:sections), we can produce the whole sheaf space by restricting and collating them relative to an open cover \mathcal{U} of the base topological space X . This is the very process of ‘sheafification’ (of a preasheaf) (Mallios, 1998a).

²⁹ In particular, $\phi_{\alpha\beta}$ can be locally expressed as the $\mathbf{A}|_{U_{\alpha\beta}}$ -isomorphism: $\phi_\alpha \circ \phi_\beta^{-1} \in \text{Aut}_{\mathbf{A}|_{U_{\alpha\beta}}}(\mathbf{A}^n|_{U_{\alpha\beta}}) = \text{GL}(n, \mathbf{A}(U_{\alpha\beta})) = \mathcal{GL}(n, \mathbf{A})(U_{\alpha\beta})$, in which expression the familiar local coordinate transition (:structure) functions appear. Hence, also the ‘natural’ structure (:gauge) group sheaf $\text{Aut}\mathcal{E} = \mathcal{GL}(n, \mathbf{A})$ of \mathcal{E} arises.

determined by $\phi_{\alpha\beta}$; write:

$$[\phi_{\alpha\beta}] \in H^1(X, \mathcal{GL}(n, \mathbf{A})) = \lim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A})) \quad (7)$$

where the \mathcal{U} s, normally assumed to be *locally finite open coverings* of X (Mallios, 1998a,b; Mallios and Raptis, 2001, 2002, 2003), constitute a *cofinal subset* of the set of all proper open covers of X .³⁰ *In toto*, we assume that $\phi_{\alpha\beta}$ encodes all the (local) information we need to determine the local quantum-particle states of the field in focus (i.e., the local sections of \mathcal{E}).

2. On the other hand, it is well known that *locally \mathcal{D} is uniquely determined by the so-called 'gauge potential' ω* , which is normally (i.e., in CDG) defined as a Lie algebra (:vector) valued 1-form. Correspondingly, in ADG ω is seen to be an element of $M_n(\Omega(U)) = M_n(\Omega)(U) = \Omega(\text{End}\mathcal{E})$,³¹ thus it is called the *local \mathbf{A} -connection matrix (ω_{ij}) of \mathcal{D} , with entries local sections of $\mathcal{E}^* = \Omega$* . In turn, this means that locally \mathcal{D} splits in the familiar way, as follows:

$$\mathcal{D} = \partial + \omega \quad (8)$$

where ∂ is the usual 'inertial' (:flat) differential³² and ω the said gauge potential. In ADG-gravity, the proper field \mathcal{D} as a whole (: 'globally') represents the *gravito-inertial field*, but locally it can be separated into its inertial (: ∂) and gravitational (: ω) parts.³³

³⁰ An assumption that in the past has proven to be very fruitful in applying ADG to the formulation of a locally finite, causal and quantal VEG (Mallios and Raptis, 2001, 2002, 2003; Raptis, 2006a,b,c). We will use it in our *K*-theoretic musings in the sequel, but provisionally we note that the direct (:inductive) limit depicted in (7) above is secured by the 'cofinality' of the set of finitary (:locally finite) open coverings of X that we choose to employ (Sorkin, 1991; Raptis and Zapatin, 2000, 2001; Raptis, 2000a,b; Mallios and Raptis, 2001, 2002, 2003; Raptis, 2006a,b).

³¹ Note that, as also mentioned earlier in footnote 9, in ADG by definition one has: $\Omega := \mathcal{E}^* := \text{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$. That is, the \mathbf{A} -module sheaf Ω of abstract differential 1-forms is dual to the vector sheaf \mathcal{E} , much like in the classical theory (:CDG of \mathcal{C}^∞ -manifolds) where differential forms (:cotangent vectors) are dual to tangent vectors, although again as noted earlier in footnote 23, in ADG the epithet '(co)tangent' is meaningless due to the manifest absence of an operative background space(time) of any kind (especially, of a base manifold).

³² In ADG, ∂ , like \mathcal{D} , is defined as a linear, Leibnizian \mathbf{K} -sheaf morphism $\partial : \mathbf{A} \rightarrow \Omega$, thus it is an instance of on \mathbf{A} -connection; albeit, a *flat* one (: $R(\partial) = 0$) (Mallios, 1998a,b; Mallios and Raptis, 2001, 2002, 2003).

³³ For more discussion on the physical meaning of this local separation of the proper field \mathcal{D} into ∂ and ω , see footnote 34 below.

Thence, the envisaged sheaf cohomological canonical quantization-type of scenario for the total ADG-fields $\mathfrak{F} = (\mathcal{E}, \mathcal{D})$ rests essentially on positing the following non-trivial abstract Heisenberg-type local commutation relations between (the characteristic forms that completely characterize) \mathcal{E} (:abstract *position* states) and \mathcal{D} (:abstract *momentum* operator). Thus, heuristically we posit:

$$[\phi_{\alpha\beta}, \partial + \omega_{ij}]_{U_{\alpha\beta}} = [\phi_{\alpha\beta}, \partial]_{U_{\alpha\beta}} + [\phi_{\alpha\beta}, \omega_{ij}]_{U_{\alpha\beta}} \quad (9)$$

stressing also that

the local commutation relations in (9) above are well defined, since they effectively *close within the noncommutative* ($n \times n$)-matrix Klein-Heisenberg algebra $\text{End}\mathcal{E}(U_{\alpha\beta}) = M_n(\mathbf{A}(U_{\alpha\beta})) = M_n(\mathbf{A})(U_{\alpha\beta})$ of the field's endomorphisms—the field's ‘noncommutative Kleinian geometry’ we mentioned at the end of the last section (:quantum field foam).

This ‘*algebraic closure*’ is in accord with the theoretical requirement we imposed earlier, namely that,

the abstract, Heisenberg-like, canonical quantum commutation relations between the two components \mathcal{E} and \mathcal{D} of the ADG-fields should not take us outside the fields, but should rather *close* within them.³⁴

Indeed, $\text{End}\mathcal{E}$ is precisely the algebra sheaf of internal/intrinsic (dynamical) self-transmutations of the (quantum particle states of the) field—by definition, the \mathcal{E} -endomorphisms in $\text{Hom}_{\mathbf{A}}(\mathcal{E}, \mathcal{E})$ (:quantum field foam). This is another aspect of the quantum dynamical autonomy of ADG-fields:

the \mathcal{E} (:abstract point-particle/position) part of the ADG-field is ‘*complementary*’, in the quantum sense of ‘complementarity’, to \mathcal{D} (:abstract field-wave/momentum). Thus, the total ADG-fields \mathfrak{F} are ‘*quantum self-dual*’ entities (Mallios and Raptis, 2003, 2005; Raptis, 2006a,c), as we anticipated in footnote 19.³⁵

Furthermore, by choosing $\phi_{ab} = \phi_{ab}^{in}$ ³⁶ so that ω is ‘gaugedaway’—i.e., by

³⁴ Here, one could envisage an abstract Heisenberg-type of algebra freely generated (locally) by ϕ (:abstract position) and ω (:abstract momentum), modulo the (local) commutation relations (9). Plainly, it is a subalgebra of $\text{End}\mathcal{E}(U)$, but deeper structural investigations on it must await a more complete and formal treatment (Raptis, 2006d).

³⁵ From our abstract perspective, the de Broglie-Schrödinger wave-particle duality is almost tautosemous with the Bohr-Heisenberg momentum-position complementarity.

³⁶ The superscript ‘*in*’ stands for ‘*inertial*’, and it represents a choice (:our choice!; see next footnote) of a local change-of-gauge $\phi_{\alpha\beta}^{in} \in \mathcal{GL}(n, \mathbf{A})_{\alpha\beta} \equiv \Gamma(U_{\alpha\beta}, \mathcal{GL}(n, \mathbf{A}))$ that would take us to a locally inertial frame of \mathcal{E} over $U_{\alpha\beta} \subset X$.

setting $\omega = 0$,³⁷ reduces (9) to (omitting the local open gauge indices/subscripts ‘ α, β ’):

$$[\phi^{in}, \partial] = \phi^{in} \circ \partial - \partial \circ \phi^{in} \tag{10}$$

Moreover, since we are sheaf cohomologically guaranteed that $\partial \circ \phi = 0$ globally, which is tantamount to the very *existence* of an **A**-connection \mathcal{D} (globally) on \mathcal{E} Mallios (1998a,b, 2005),³⁸ (10) further reduces to:

$$[\phi^{in}, \partial] = \phi^{in} \circ \partial \tag{11}$$

Now, a heuristic physical interpretation can be given to (11) if we consider its effect (:action) on a local section $s \in \mathcal{E}_{\alpha\beta} := \mathcal{E}(U_{\alpha\beta}) \equiv \mathcal{E}|_{U_{\alpha\beta}}$:

$$[\phi^{in}, \partial](s) = (\phi^{in} \circ \partial)(s) = \phi^{in}(\partial s) \tag{12}$$

Equation (12) designates the inertial dynamical action of \mathcal{D} (i.e., the action of its locally flat, inertial part ∂) on (an arbitrary) s , followed by the gauge transformation of ∂s to an inertial frame $e_{in}^{U_{\alpha\beta}} \subset \mathcal{E}_{\alpha\beta}$ ‘*covarying*’ with the inertio-gravitational field. It expresses what happens to a ‘vacuum graviton state’ s when it is first acted upon³⁹ by the inertial part of the proper ADG-gravitational field \mathcal{D} and then⁴⁰ to an inertial frame that in a sense ‘*covaries*’ with the said inertial change ∂ of s .

Perhaps one can get a more adventurous (meta)physical insight into (12) by defining the *uncertainty operator* \mathfrak{U} as

$$\mathfrak{U} := \phi^{in} \circ \partial \in \text{End}\mathcal{E} \tag{13}$$

³⁷This is an analogue of the Equivalence Principle (EP) of GR in ADG-gravity, corresponding to the local passage to an ‘inertial frame’ (:one ‘covarying’ with the gravitational field; e.g., recall Einstein’s free falling elevator *gedanken* experiment) in which the curved gravito-inertial \mathcal{D} in (8) reduces to its flat ‘inertial’ **A**-connection part ∂ (Mallios, 1998a,b; Mallios and Raptis, 2001, 2002, 2003). This just reflects the well known fact that *GR is locally SR*, or conversely, that when SR is localized (i.e., ‘gauged’ over the base spacetime manifold) it produces GR (equivalently, the curved Lorentzian spacetime manifold of GR is locally the flat Minkowski space of SR). *In summa*, gravity (: ω) has been locally gauged away, and what we are left with is the inertial action ∂ of the ADG-gravitational field \mathcal{D} . It must be also stressed here that the choice of a locally inertial frame, like all gauge choices, is an externally imposed constraint in the theory—‘externally’, meaning that it is *we*, the external (to the field) experimenters/theoreticians (‘observers’) that we impose such constraints on the field (i.e., we make choices about what aspects of the field we would like to single out and, ultimately, observe/study).

³⁸This essentially corresponds to the fact that the coordinate 1-cocycle $\phi_{\alpha\beta} \in Z^1(\mathcal{U}, \Omega)$ is actually a *coboundary* (a closed form), belonging to the zero cohomology class $[\partial\phi_{\alpha\beta}] = 0 \in H^1(X, M_n(\Omega))$, which in turn guarantees the existence of an **A**-connection on \mathcal{E} as the so-called *Atiyah class* \mathfrak{a} of \mathcal{E} vanishes (: $\mathfrak{a}(\mathcal{E}) := [\partial\phi_{\alpha\beta}] = 0$) Mallios (1998a,b, 2005).

³⁹Recall that we are considering only *vacuum* gravity, in which the non-linear gravitational field ‘couples’ solely to itself(!)

⁴⁰The sequential language used here should not be interpreted in an temporal-operational sense—as it were, as ‘operations carried out sequentially in time’.

and by delimiting all the quantum-particle (:abstract position) states of the field (:local sections of \mathcal{E}) that are annihilated by it. Intuitively, these are formally the local ‘classical-inertial’ states

$$\mathcal{E}_U^{cl} := \text{span}_{\mathbf{K}}\{s \in \mathcal{E}(U) : \mathfrak{U}(s) = 0\} =: \ker(\mathfrak{U}) \quad (14)$$

for which the abstract sheaf cohomological Heisenberg uncertainty relations (10) vanish. Plainly, $\mathcal{E}^{cl}(U)$ is a \mathbf{K} -linear subspace of $\mathcal{E}(U)$ —the kernel of \mathfrak{U} .

On top of the above, intuitively it makes sense to assume that \mathfrak{U} is a ‘projector’—a primitive idempotent (:projection operator) locally in $\text{End}\mathcal{E}$ (i.e., in $M_n(\mathbf{A}(U))$)—since the ‘gedanken’ operation of ‘*inertially covarying with a chosen local inertial frame*’ must arguably be idempotent.⁴¹ This means that $\mathfrak{U}^2 = \mathfrak{U}$, so that \mathfrak{U} separates (chooses or projects out) the ‘classical’ ($\text{eigen}_0(\mathfrak{U}) \equiv \ker(\mathfrak{U})$) from the quantum ($\text{eigen}_1(\mathfrak{U})$) local quantum gravito-inertial states. A formal reason why we chose \mathfrak{U} to be a projection operator will become transparent in our K -theoretic musings a little bit later.

Finally, we would like to ask *en passant* here the following highly speculative question:

Could the generation/emergence of (inertio-gravitational) mass be somehow accounted for by a (spontaneous) symmetry breaking-type of mechanism, whereby, the dynamical automorphism group $\text{Aut}\mathcal{E}$ of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ reduces to its subgroup that leaves $\ker(\mathfrak{U})$ invariant? Alternatively intuited, could the emergence of inertio-gravitational mass be thought of as the result of some kind of ‘quantum anomaly’ of 3rd-quantized VEG?⁴²

Functoriality: the quintessence of 3rd-quantization

We commence this subsection by noting that in ADG-gravity the notion of *functoriality* plays a very significant role and has a very precise physical interpretation in the theory, with significant, we believe, implications for certain current and future QG research issues:

⁴¹ After all, ‘*inertially covarying the inertial state leaves it inertially covariant*’. Or, to use a famous Einstein ‘gedanken metaphor’: ‘*jumping on a light-ray (in order to ride it) twice, simply leaves you riding it*’ (!)

⁴² The epithet ‘quantum’ adjoined to ‘anomaly’ is intended to distinguish the effect intuited above from the usual anomalies. A ‘quantum anomaly’ is the ‘converse’ of an anomaly in the usual sense, in that what was a symmetry of the *quantum* theory (:an element of $\text{Aut}\mathcal{E}$ in our case) ceases to be a symmetry of the ‘classical domain’ of our theory (: $\ker(\mathfrak{U})$). Let it be stressed that the emergence of gravito-inertial ‘mass’ in the sense intuited here has a truly relational (:algebraic) and ‘global’ flavour reminiscent of Mach’s ideas: ‘global’ gravitational field symmetries in $\text{Aut}\mathcal{E}$ are locally reduced to inertial ones, and sheaf theory’s ability to interplay between local and global comes in handy in this respect (Mallios and Raptis, 2005). (See further remarks below on how sheaf theory allows us to go from ‘local’ to ‘global’, and vice versa.)

In our scheme, functoriality pertains to **A**-functoriality of the VEG and the FYM dynamics (1)–(3). This means that the geometrically (pre)quantized, 2nd-quantized and, ultimately, 3rd-quantized VEG and FYM dynamical equations are not 'perturbed' at all by our acts of coordinatization (: 'measurements') encoded in the **A** that we choose up-front to employ as structure sheaf of generalized arithmetics (:coordinates).⁴³ Plainly, this is so, because both the VEG curvature and the FYM field strength involved in (1) and (3) respectively are **A**-morphisms (:alias, $\otimes_{\mathbf{A}}$ -tensors).⁴⁴

Concerning ADG-VEG in particular, **A**-functoriality is an abstract version of the PGC of the manifold based GR (Mallios and Raptis, 2003, 2005; Raptis, 2006a,b,c).

At this point, before we go into discussing functoriality *vis-à-vis* prequantization, 1st, 2nd and 3rd-quantization, we would like to dwell for a while on how **A**-functoriality in ADG-VEG may evade two apparently insurmountable (by CDG-means) problems in classical and quantum GR.

The PGC of GR, when mathematically modelled after the $\text{Diff}(M)$ group of the C^∞ -smooth spacetime manifold based (and, *in extenso*, CDG-founded) GR, creates serious problems in both the classical and the quantum theory, briefly as follows:

1. Traditionally, gravitational singularities are supposed to be a problem of the classical field theory of gravity (:GR). There, the PGC appears to come into conflict with the very existence of singularities to the extent that until today there is no unanimous agreement on (or anyway, a clear-cut definition of) what is a singularity in the theory (Geroch, 1968; Clarke, 1986, 1993). Granted that singularities are built into the differential manifold that we assume up-front to model 'spacetime' in GR,⁴⁵ it is hardly surprising, especially in view of S1, that the usual manifold based Analysis comes short of resolving or evading them completely (Clarke, 1986,

⁴³This choice of ours may be understood as follows: it is we that choose an **A** to coordinatize the dynamical connection field proper \mathcal{D} and then represent it as acting on the vector sheaf \mathcal{E} . The latter, which is locally a finite power of **A**, is the representation (:associated) sheaf of the group sheaf $\text{Aut}\mathcal{E}$ of dynamical self-transmutations (:automorphisms) of the field, and it can thus be regarded as the 'carrier space' (for the action) of \mathcal{D} .

⁴⁴Where $\otimes_{\mathbf{A}}$ is the homological tensor product functor between the relevant categories involved (mainly, the category of **A**-modules). $\otimes_{\mathbf{A}}$ -tensors are the 'geometrical objects' in our theory as, in a Kleinian sense, they are left invariant under $\text{Aut}_{\mathbf{A}}\mathcal{E}$.

⁴⁵That is to say, singularities are *loci* in the spacetime continuum M in the vicinity of which some smooth function component of the smooth metric $\otimes_{\mathbf{A}=C_M^\infty}$ -tensor solution of the Einstein equations (: $g_{\mu\nu}$) appears to blow up uncontrollably without bound, and the associated differential equations of Einstein appear to 'break down' in one way or another (Clarke, 1986, 1993; Raptis, 2006a; Mallios and Raptis, 2005). Thus, from the ADG-theoretic perspective, smooth gravitational singularities are inherent in the coordinate structure sheaf $\mathbf{A} \equiv C_X^\infty$ that we assume up-front in GR, which is tantamount to our *a priori* assumption of a base differential manifold M in that CDG-based classical field theory of gravity (Gel'fand duality).

1993; Raptis, 2006a; Mallios and Raptis, 2005). For, to tackle singularities, we are in the first place using a Differential Calculus (:CDG) that is vitally dependent on a background smooth manifold that is carrying the very singularities we are trying to resolve. There appears to be no way out of this vicious circle as long as we insist on doing GR by manifold based CDG-means. To paraphrase and extend a well known quote of Peter Bergmann:

it is not surprising that GR “*carries the seeds of its own destruction*” Bergmann (1979) in the form of singularities, since, the manifold based CDG employed to develop the classical field theory of gravity already carries in its foundation, its soil as it were ($:M \leftrightarrow C^\infty(M)$)⁴⁶ those very singular germs (pun intended).

Mutatis mutandis then for the structure group of the underlying spacetime manifold: it is not surprising that $\text{Diff}(M)$ clashes with our attempts at giving a clear-cut ‘definition’ of gravitational singularities. Plainly, at the basis of the aforesaid vicious self-referential problem lies M , so that what behooves us is to ask whether there is an alternative differential calculus—one that does not depend at all on a base manifold, yet one that can reproduce all the constructions and results of the manifold based CDG, if we wished to—by which we can view (and actually do!) gravity as a field theory.

Of course, for us this is a rhetorical question since we have ADG. By ADG-means we have completely evaded singularities of all sorts (Mallios and Rosinger, 1999, 2001, 2002; Mallios, 2001; Raptis, 2006a; Mallios and Raptis, 2005), on the one hand by doing away with a base differential spacetime manifold, while on the other, by ‘absorbing’ singularities in the structure sheaf \mathbf{A} of generalized arithmetics and by formulating the VEG differential equations *functorially* in terms of \mathbf{A} -sheaf morphisms. In this way singularities are *not* seen to be sites where the Einstein equations break down differential geometrically speaking, or where any sort of unphysical infinity (in the usual analytical sense) is involved. This must be attributed simply to the fact that ADG divests Calculus from, to use another famous quote now of George Birkhoff (1938), “*the glittering trappings of Analysis*”—arguably, our being trapped into the aforementioned vicious circle reflecting our theoretical ‘imprisonment’ into the background (spacetime) manifold that we assumed in the first place(!) By breaking free from the background spacetime manifold M , we totally bypass singularities, while the PGC of GR ceases to be represented by $\text{Diff}(M)$,

⁴⁶Recall Gel’fand duality: a differential manifold M is the topological algebra $C^\infty(M)$ of smooth functions on it Mallios (1986).

but purely algebraico-categorically, by $\text{Aut}_{\mathbf{A}}\mathcal{E}$ —the ADG-gravitational field’s ‘autosymmetries’.

In addition to the above, we note that the \mathbf{A} -functoriality of the ADG VEG field dynamics, which corresponds to an abstract version of the PGC of the manifold and CDG-based GR, can be readily generalized further by categorical means to what has been coined elsewhere the *Principle of Algebraic Relativity of Differentiability* (PAR \mathcal{D}) (Raptis, 2006a; Mallios and Raptis, 2005; Raptis, 2006b,c),⁴⁷ as follows:

Since, from the ADG-theoretic perspective, *all differential geometry boils down to \mathbf{A}* (Mallios, 1998a,b, 2005; Mallios and Raptis, 2003, 2005; Raptis, 2006a,b)—i.e., all differential geometry (indeed, the entire *aufbau* of ADG) rests on the algebra (sheaf) of ‘*differentiable functions*’ that we assume/employ up-front (as coordinates) in the theory⁴⁸—while at the same time the ADG-gravitational dynamics is \mathbf{A} -functorial, any change in (*our* choice of) structure sheaf \mathbf{A} ⁴⁹ should not affect (i.e., at least it should leave ‘form-invariant’) the ADG-VEG field dynamics (1).

Categorically speaking, this gives rise to a *natural transformation-type of functors* (MacLane and Moerdijk, 1992) between the categories involved, which can be depicted short-handedly by the following commutative diagram which we borrow almost intact from Mallios and Raptis (2005) and Raptis (2006b):

with the $\mathcal{N}_{\mathbf{A}}$ functor above designating a change in (*our* choice of) structure sheaf of generalized coordinates—from, say, \mathbf{A}_1 that might have been chosen initially, to \mathbf{A}_2 —and, as a result, from vector sheaf \mathcal{E}_1 to \mathcal{E}_2 . In turn, $\mathcal{N}_{\mathbf{A}}$ induces an ‘adjoint’ functor $\mathcal{N}_{\mathcal{D}}$, which takes us from the $\otimes_{\mathbf{A}_1}$ -functorial vacuum Einstein equations holding on \mathcal{E}_1 , to similarly $\otimes_{\mathbf{A}_2}$ -functorial VEG equations holding on \mathcal{E}_2 . Clearly, the pair $(\mathcal{N}_{\mathbf{A}}, \mathcal{N}_{\mathcal{D}})$ of adjoint functor-type of maps above leave the ADG-VEG equations form-invariant, and have been coined in the past ‘*differential geomet-*

⁴⁷This is a generalization of how the PGC of GR was seen to be a direct consequence of Einstein’s original Principle of Relativity maintaining that the field law of gravity should hold in any coordinate system Einstein (1952, 1956).

⁴⁸This ‘aphorism’, i.e., that all DG rests on (*our* choice of) \mathbf{A} , hinges on Mallios’ fundamental observation that the notion of differential ∂ (*viz.* connection \mathcal{D})—arguably, *the* basic concept with which one can actually talk about (and do!) *differential* geometry—is vitally dependent on what (algebras of) functions we declare and assume up-front as being ‘differentiable’. These functions then provide us with the differential operators we need in order to do DG. Recall for example the very definitions of ∂ and \mathcal{D} in ADG: $\partial : \mathbf{A} \rightarrow \Omega$ and $\mathcal{D} : \mathcal{E} \rightarrow \Omega$ (Mallios, 1998a,b; Mallios and Raptis, 2003), both of which have effectively \mathbf{A} as their domain of definition.

⁴⁹Essentially, any choice of what *we* (‘arbitrarily’) perceive (and define!) as ‘*differentiable functions*’. Such changes are entirely up to the theorist (and, *in extenso*, to the observer or experimenter), if for instance she wishes to consider another, more suitable to the physical situation/problem she encounters, algebra of coordinate functions in which to absorb a singularity that she confronts.

$$\begin{array}{ccc}
 \mathbf{A}_1(: \mathcal{E}_1 \stackrel{\text{loc.}}{\cong} \mathbf{A}_1^n) & \xrightarrow{\otimes_{\mathbf{A}_1}} & \mathcal{R}(\mathcal{E}_1) = 0 \\
 \mathcal{N}_{\mathbf{A}} \downarrow & & \downarrow \mathcal{N}_{\mathcal{D}} \\
 \mathbf{A}_2(: \mathcal{E}_2 \stackrel{\text{loc.}}{\cong} \mathbf{A}_2^n) & \xrightarrow{\otimes_{\mathbf{A}_2}} & \mathcal{R}(\mathcal{E}_2) = 0
 \end{array} \tag{15}$$

ric morphisms' Raptis (2006b).⁵⁰ Thus, the most general ADG-theoretic expression of the PGC of GR is the following:

The VEG ADG-field equations (1) are left invariant under \mathbf{A} -geometric morphisms.

Furthermore, on the PARD and the most general ADG-expression of the PGC above rests the following *Principle of ADG-Field Realism*(PFR) (Mallios and Raptis, 2005; Raptis, 2006a,b,c):

No matter what \mathbf{A} we use to (differential geometrically) represent the gravitational field dynamics,⁵¹ the latter remains unaffected (:'unperturbed') by our (generalized) 'coordinate measurements' (in \mathbf{A}). The connection field proper \mathcal{D} exists 'out there', independently of our coordinatizations (and differential geometric representations) by \mathbf{A} (and, *in extenso*, by \mathcal{E} which is locally of the form \mathbf{A}^n). Moreover, since, as noted earlier, if there is any 'spacetime' in our theory, it is encoded in \mathbf{A} , the ADG-gravitational field \mathcal{D} and the VEG law (1) that it defines differential geometrically as a differential equation proper, 'sees through' (i.e., it remains unaffected by) it. The field \mathcal{D} , and the law (1) that it defines, is oblivious to our own coordinatizations/coordinate measurements in \mathbf{A} (\mathbf{A} -functoriality) and therefore also to the 'spacetime geometry' encoded in the latter (by Gel'fand duality).

2. In various, both canonical and covariant, gravitational field quantization scenaria, the background M and its $\text{Diff}(M)$ structure (:'symmetry') group create some formidable problems. Consider for example the situation in canonical QGR approached via, say, LQG: there, since the background spacetime continuum is retained, one regards gravity as an external spacetime constrained gauge theory; hence, one has to account for the spatial

⁵⁰The ADG-theoretic analogue of the fundamental notion of *geometric morphism* in sheaf and topos theory MacLane and Moerdijk (1992). Moreover, since, as noted above, from the ADG-theoretic perspective all differential geometry is based on \mathbf{A} , differential geometric morphisms may be equivalently called \mathbf{A} -*geometric morphisms*.

⁵¹Effectively, *our* choice of representation sheaf \mathcal{E} on which the connection field \mathcal{D} acts.

and temporal diffeomorphism constraints in the quantum theory (:primary constraints in a Dirac-type of quantization). This results in notorious problems that CQGR has to resolve in order to proceed, such as the problem of defining meaningful gravitational $\text{Diff}(M)$ -invariant quantum observables,⁵² the associated problem of finding the physical Hilbert space of states and its inner product, as well as the (in)famous problem of time (Isham, 1993; Kuchař, 1993; Torre, 1994). Let alone that, on top of all this, one has to try to preserve manifest (general) covariance in the quantum theory when *ab initio* the canonical formalism appears to mandate a $3 + 1$ space-time split (:a foliation of the base manifold into space-like hypersurfaces) in order for it to make any sense at all (i.e., to be able to make sense of canonical, ‘equal-time’ commutation relations between canonically conjugate gravitational field variables).

Covariant (:path-integral) quantization of gravity scenaria also encounter challenging obstacles due to the presence of the background manifold and its $\text{Diff}(M)$ structure group. The Ashtekar new connection variables 1st -order formalism, for example Ashtekar (1986), significantly simplified the constraints in CQGR and revealed the ‘innate’ gauge-theoretic character of gravity; albeit, by retaining a base smooth manifold. In particular, it showed us that the physical configuration space in the theory is the $\text{Diff}(M)$ -moduli space of (gauge) equivalent spin-Lorentzian connections. It follows that a possible quantization scenario for gravity could involve a functional integral over the said moduli space. Thus, an integro-differential calculus on the aforesaid affine space of smooth connections on a manifold should be developed (Ashtekar and Lewandowski, 1995a,b), and the ever-presence of the infinite-dimensional $\text{Diff}(M)$ group on the background does not make life any easier. In particular, one should search for $\text{Diff}(M)$ -invariant Faddeev-Popov-type of measures on the moduli space of gauge equivalent connections in order to implement the said functional integral—a daunting task indeed (Baez, 1994a,b, 1997).

At the end of the last section we are going to return briefly to these QG issues and discuss briefly our ADG-stance against them. Now however, we would like to go back and dwell a bit on the issue of functoriality *vis-à-vis* pre-, 1st-, 2nd- and 3rd-quantization.

Following (the intro to) (Mallios, 2004), we note that in much the same way that the principal aim of geometric (pre)quantization and 2nd-quantization is to bypass 1st-quantization and give directly a quantum description of relativistic fields (i.e., without needing first to quantize the corresponding classical mechanical

⁵² Especially in vacuum Einstein gravity Torre (1993, 1994).

particle or field theory),⁵³ one can regard as the principal reason for the ADG-based 3rd-quantization as a need for

a direct quantum description of the ADG-fields ‘in themselves’,⁵⁴ without any reference to or dependence on an external (:background to the fields) spacetime manifold.

Since both the usual geometric (pre)quantization and second quantization scenaria are essentially rooted on a base spacetime manifold for their differential geometric formulation in terms of CDG (Brylinski, 1993; Bandyopadhyay, 1996; Woodhouse, 1997; Echeverria-Enriquez *et al.*, 1999; Isidro, 2004), the ADG-based 3rd-quantization may be seen as an extension and generalization of both,⁵⁵ hence the epithet ‘third’ in order to distinguish between them at least nominally. An important consequence of this is that while it is meaningful in 2nd-quantized GR (e.g., QGR approached via LQG, which is based on the manifold dependent Ashtekar formalism) to talk about ‘*spacetime continuum quantization*’,⁵⁶ in 3rd-quantized ADG-gravity it is simply meaningless, since no spacetime manifold, external to the ADG-gravitational field, is involved (i.e., *a priori* assumed in the theory).⁵⁷

On the other hand, it is well known that geometric prequantization and second quantization are manifestly functorial procedures. In what follows we would like to ponder a bit on the functoriality of 3rd-quantization and its physical significance in addition to our comments on **A**-functoriality *vis-à-vis* gravitational singularities and QG issues above.

⁵³ In this respect, we may recall from Mallios (2004) the following two contrasting quotes found in Woodhouse (1997) and Goldstein (1950), respectively: “*to find a quantum model of ... an elementary relativistic particle it is unnecessary ... to quantize [first] the corresponding classical system*” and “*... to quantize a field, we have first to describe it in the language of mechanics*”. In ADG-field theory, where, following Einstein Einstein (1956), “*[In the theory of relativity,] the field is an independent, not further reducible fundamental concept...[so that] the theory presupposes the independence of the field concept*”, it is plain that we ‘take sides’ with the geometric quantization camp (see also Zeh, 2003 for more, but slightly differently motivated, arguments against 1st-quantization of a classical mechanical/field theory).

⁵⁴ This is an autonomous, self-quantization in accord with the quantum self-duality of the ADG-fields we saw earlier in this section (Mallios and Raptis, 2005).

⁵⁵ In line with the general fact that ADG is a significant abstraction and generalization of CDG.

⁵⁶ Something that, as noted earlier, is of great import in avoiding/resolving gravitational spacetime singularities in LQG for example (Husain and Winkler, 2004; Modesto, 2004).

⁵⁷ For instance, as noted before, the evasion of gravitational singularities in ADG-gravity is secured by the **A**-functoriality of the ADG-gravitational dynamics (1) as singularities of all kinds (even dense, non-linear distributional ones never encountered before in the mathematics or physics literature!) are absorbed in **A** (Mallios and Rosinger, 1999, 2001, 2002; Mallios and Raptis, 2005; Raptis, 2006a).

Half, first and second quantization

Broadly speaking, *prequantization*, or what we here coin ‘*half quantization*’, pertains to a formal mathematical procedure which establishes a *correspondence* between the classical description of a physical system and a quantum description of the same system. Given the standard mathematical model of the kinematical space of a classical mechanical system as a smooth phase space M (:a differential manifold) endowed with a symplectic structure ω on it (:a symplectic manifold, write $S = (M, \omega)$), together with a Hamiltonian function H generating its smooth dynamical (:time) evolution, $\frac{1}{2}$ -quantization corresponds it to a Hilbert space \mathcal{H} in such a way that the transformations of M preserving ω (:the canonical or symplectic maps) are mapped to unitary operators on \mathcal{H} , which by definition preserve \mathcal{H} ’s inner product (:isometries).⁵⁸ In *category-theoretic* terms,

prequantization is a functor from the category of symplectomanifolds and canonical morphisms, to the category of Hilbert spaces and unitary operators on them.

On the other hand, it is also well known that *first quantization*, unlike *prequantization*, is *not* a functorial procedure. By 1st-quantization one means a correspondence, like *prequantization*, between the aforesaid symplectic and Hilbert space categories which furthermore carries a one-parameter group of canonical transformations generated by a *positive* H , to a one-parameter group of unitaries generated by a *positive* Hamiltonian operator \hat{H} . The bottom line here is that

first quantization is not functorial, because energy-positivity is not preserved in transit from the classical to the quantum description.

However, if one has established a single-quantum (:particle) description of a physical system, one can pass *functorially* to a many-particle one (e.g., a quantum field) by the process of *second quantization*. Briefly, starting from the single-quantum Hilbert space \mathcal{H} above, and depending on the spin of the particle-quanta of the fields considered, one takes completely symmetric or antisymmetric tensor products of any number of identical copies of \mathcal{H} , which when directly summed and completed yields the so-called Fock state space in which free quanta of the corresponding boson and fermion fields are supposed to live. Thus, the *2nd-quantization functor* is from the Hilbert category to itself (appropriately tensored),⁵⁹ and it can be easily seen to be positivity preserving.

To summarize, while *prequantization* and *second quantization* are functorial constructions (procedures or correspondences), *first quantization* is not. Actually, as noted above, it is the *raison d’être et de faire* of *geometric (pre)quantization* to

⁵⁸ Usually one assumes \mathcal{H} to be $L^2(M)$ —the Hilbert space of smooth (\mathbb{C} -valued) square integrable functions on M relative to the standard Liouville measure μ_M on the latter, defining a Hermitian inner product: $\langle \phi | \psi \rangle := \int \phi^* \psi d\mu_M$, with ϕ^* the complex conjugate of ϕ .

⁵⁹ That is, the Hilbert category is augmented by the usual tensor product \otimes , hence it is regarded as a tensor monoidal category.

bypass completely 1st-quantization and describe directly 2nd-quantized (:quantum) fields, without recourse to a classical mechanical particle or field system which has to be quantized first.⁶⁰

ADG in second and geometric (pre)quantization: further general remarks

The aforesaid bypass of 1st-quantization by geometric quantization in order to arrive directly at a quantum description of fields suits perfectly ADG, since in the latter the basic objects—its fundamental building blocks or *ur*-elements so to speak—are ADG-fields of the $(\mathcal{E}, \mathcal{D})$ kind. Thus, in ADG-field theory, in keeping with the terminology and technical machinery of the manifold and CDG-based geometric (pre)quantization (Brylinski, 1993; Bandyopadhyay, 1996; Woodhouse, 1997; Echeverria-Enriquez *et al.*, 1999; Isidro, 2004), albeit in a manifestly background manifold independent way as befits ADG, the epithet ‘geometric’ to ‘(pre)quantization’ pertains to the identification of the ω of a symplectic manifold—a closed differential 2-form on M —to the curvature $R(\mathcal{D})$ of a connection field \mathcal{D} on a (Hermitian) Hilbertian representation vector sheaf \mathcal{E} . The latter can be regarded, via an ADG-theoretic generalization of the celebrated Serre–Swan theorem (Mallios, 2004, 2005) originally motivated by some arguments of Selesnick in the context of 2nd-quantization (Selesnick, 1983), as a *free, finitely generated projective \mathbf{A} -module, whose (Hilbert space) stalks represent the (pre)quantum state spaces of the ADG-field systems in focus* (Mallios, 1999, 2004, 2005). In turn, (the curvature of the connection on) \mathcal{E} obeys some sort of *quantization condition* (e.g., Weil’s integrality condition), which is instrumental in classifying sheaf cohomologically the vector sheaves involved (e.g., Chern-Weil theorem, Chern characteristic classes, the Picard group) (Mallios, 1998b, 1999, 2005). Moreover, as we highlighted it earlier in Section 2, from a 2nd-quantization vantage the local sections of the Hilbertian \mathcal{E} s represent *local quantum particle states of the corresponding fields*, while also an ADG-theoretic version of the spin-statistics connection comes to identify *local free boson states with local sections of line sheaves*, while *local particle states of fermionic fields correspond to local sections of Grassmannian vector sheaves of finite rank greater than 1*.

All this is well done and dusted; however, here we would like to make a couple of additional scholia in the light of 3rd-quantization presently proposed and its \mathbf{A} -functoriality:

⁶⁰ Indeed, on general philosophical grounds, it is *unnatural* to suppose that there is a classical/quantum dichotomy in Nature (pun intended). Quantum theory is the fundamental description, while all classical ones are coarse approximations (:effective descriptions) of the fundamental quantum one (Mallios and Raptis, 2005; Raptis, 2006a,c). To paraphrase Finkelstein: “*all is quantum; anything that appears to be classical has not been resolved yet into its quantum elements*” (Finkelstein, 1996).

1. Quantum fields are traditionally regarded as special relativistic entities,⁶¹ hence their quantum particle states have been modelled after irreducible representations of the Poincaré group *à-la* Wigner. At the same time, what has for many years stymied efforts to genuinely unite relativity with quantum theory in a finitistic setting is the fact that, because the Lorentz group is non-compact, there are no *finite-dimensional* unitary representations of it Haag (1996).⁶² Yet, the reader must have already observed that our representation vector (:Hilbert **A**-module) sheaves \mathcal{E} are of finite rank, and they constitute unitary representation spaces of the ‘internal’ (:gauge) symmetry groups, which are of course compact Mallios (2004, 2005). There is no discrepancy here: 3rd-quantum field theory does not involve any base spacetime at all—be it flat Minkowski space or a curved gravitational background; hence, we do not have to account for a potential conflict between finite-dimensionality of (unitary) particle representations and the Lorentz group.⁶³ *In summa, no background spacetime, no external spacetime symmetries; and all the symmetries of the ADG-fields are ‘internal’ (:gauge)—another positive feature of the 3rd-gauge 3rd-quantum ADG-fields’ autonomy.*
2. As noted earlier, the central result of geometric (pre)quantization is the identification of the symplectic form ω on the C^∞ -manifold phase space of a physical system with the curvature of a connection (on the same manifold). The aforementioned **A**-functoriality of the ADG-VEG dynamics (1) comes in handy here since the latter is an expression involving the geometrically (pre)quantized curvature R of the ADG-gravitational field proper \mathcal{D} , and $R(\mathcal{D})$ is an $\otimes_{\mathbf{A}}$ -tensor (:an **A**-morphism). Precisely in this sense we maintained in section 2 that we already possess a geometrically (pre)quantized—and, in view of (Mallios, 2004), a 2nd-quantized—vacuum gravitational (and, *in extenso*, free YM) dynamics. Moreover, this dynamics is manifestly generally covariant (in the gen-

⁶¹ After all, QFT is supposed to be a unison of SR and QM (:quantum fields over flat Minkowski space).

⁶² In Finkelstein (1969) for example, this shortcoming was diagnosed early in building the ‘Space-Time Code’, or subsequently in developing the quantum net dynamics (Finkelstein, 1988, 1989), and Finkelstein opted for sacrificing quantum mechanical unitarity (but preserve algebraic finiteness!), because it is a non-local notion (as it involves an *integral* over all space). Furthermore, for quantum fields over a curved spacetime manifold (in case for instance one wished to apply the QFTheoretic formalism ‘blindfoldedly’ to gravity in a semi-classical manner) the situation is even worse, since there are inequivalent (unitary) representations of the gauged (:spacetime manifold localized) Lorentz group (Fulling, 1989).

⁶³ In any case, it has been argued for a long time now whether exact (local) Lorentz invariance would survive in the finitistic QG domain (:below its Planck length ‘cut-off’). Especially in the supposedly inherently discrete setting of Sorkin’s causal set theory (Sorkin, 1995, 1997), the issue of whether Lorentz invariance should be preserved or given up for good has recently become a caustic one, with current tendencies leaning towards abandoning it (Henson, 2006a,b; Bombelli *et al.*, 2006).

eralized ADG-sense of general covariance involving $\text{Aut}\mathcal{E}$), and it is of course also left form-invariant under \mathbf{A} -geometric morphisms. Finally, it is straightforward to see that the sheaf cohomological local Heisenberg uncertainty relations (9) defining 3rd-quantization remain invariant under \mathbf{A} -geometric morphisms, something that further vindicates the ‘universal-ity’ (‘functoriality’) of 3rd-quantization.

3. The final remark we wish to make here is a twofold, quite general one, bearing historical/methodological undertones and going at the heart of ADG *vis-à-vis* applications of differential geometry to the quantum (gauge field) domain. First, we must highlight the use of *sheaves* (and sheaf cohomology) instead of *fiber bundles* in gauge field theory that ADG advocates. Fiber bundles is *the* mathematical theory currently used in (applications of differential geometric ideas to quantum) gauge (field) theories. However, it has been recently felt that fiber bundles come short in modelling what’s ‘really’ happening in the quantum field and, *in extenso*, in the QG regime, and should thus be replaced by sheaves. We let Haag (1996) and Stachel (1987, 2003) do the talking here:

“...**Germ**s. We may take it as the central message of Quantum Field Theory that all information characterizing the theory is strictly local i.e. expressed in the structure of the theory in an arbitrarily small neighborhood of a point.⁶⁴ For instance in the traditional approach the theory is characterized by a Lagrangean density. Since the quantities associated with a point are very singular objects, it is advisable to consider neighborhoods. This means that instead of a fiber bundle one has to work with a sheaf. The needed information consists then of two parts: first the description of the germs, secondly the rules for joining the germs to obtain the theory in a finite region...”

and

“...However, the fibre bundle approach clearly does not solve the second problem discussed in the previous section.⁶⁵ The topology of the base manifold is given *a priori*, so that a different fibre bundle must be introduced *a posteriori* for each topologically distinct class of solutions. The process of finding the global topology cannot be formalized within the fibre bundle approach. It appears that sheaf cohomology theory is the appropriate mathematical theory for dealing with the problem of going from local to global solutions...” Stachel (1987)

⁶⁴ Our emphasis.

⁶⁵ That is, not fixing up-front the global topology of the manifold, and globalizing a local solution to the Einstein equations—*in toto*, (analytically) extending a local solution (a local region where the law holds) to a global one (the total spacetime manifold where the law is valid).

also

“... sheaf theory might be the appropriate mathematical tool to handle the problem⁶⁶ in general relativity. As far as I know, no one has followed up on this suggestion, and my own recent efforts have been stymied by the circumstance that all treatments of sheaf theory that I know assume an underlying manifold...” Stachel (2003)

Indeed, a virtue of sheaf theory is that it ‘naturally’ effectuates easily (virtually by the very definition of a sheaf) the much desired transition from local to global (or ‘micro’ to ‘macro’), and by its very definition models (dynamically) ‘variable structures’.⁶⁷ We should also stress here that sheaf theory, at least as it has been developed and used in ADG, does not involve at all any base manifold like the sheaves in Minkowski space (QFT) or over a general curved spacetime manifold (QG) that Haag (implicitly) and Stachel (explicitly) allude to in the extracts above.

It is about time theoretical physicists (in particular, quantum field theorists) broke from the mold of bundles and became acquainted with the rudiments of sheaf theory. Yet, one has to appreciate the hitherto unprecedented in the mathematics literature use of sheaf theory, in all its generality, power and resourcefulness, in *differential* geometry that ADG has brought about. That one can do differential geometry purely algebraically, independently of any notion of ‘smoothness’ in the standard sense (of employing a background C^∞ -manifold to ‘mediate’ our differential calculus) is indeed a feat of ADG that could have enormous implications (and applications) in current QG and quantum gauge theory research.

The second pyche of ADG we would like to highlight is the use of general, possibly *non-normed*, topological algebras in the quantum (field) regime. Briefly, ever since the von Neumann quantum axiomatics in Hilbert space, quantum (field) theorists have (pre)occupied themselves with the study of von Neumann and C^* -algebras, the ‘canonical’ example being the non- C^* -algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on Hilbert space after which algebras of quantum mechanical observables are usually modelled (Bratteli, 1979; Haag, 1996). On the other hand, the archetypical example of a non-normable (:non-Banachable) topological algebra is $C^\infty(M)$ —in fact, the *only* algebra we have used so far to do *differential* geometry (:CDG on manifolds). Admittedly, one could try to retain non- C^* -ness and try to develop a differential geometry based on such algebras, as in the ‘Noncommutative Calculus’ of Connes (1994). Yet, one could

⁶⁶The problem noted above: going (e.g., extending a solution to the field equations) from ‘local’ to ‘global’.

⁶⁷Briefly, *localization (of a structure) is gauging it, and gauging (a structure) is tantamount to making it (dynamically) variable*—i.e., endowing it with a dynamical connection field.

object even ‘in principle’ to such an endeavor by observing that operators of quantum physical interest such as ‘position’ are essentially *unbounded*, while at the same time, the reason behind the use of commutative algebras as structure sheaf of generalized coordinates (as it is the case in ADG) is Bohr’s correspondence principle. Namely that, while quantum mechanical actions are noncommutative, *our* measurements (ultimately, our geometrical representations and interpretations!) of them are essentially commutative.⁶⁸ The bottom line here is, to paraphrase Borchers this time, that physicists should break free from Banach algebras (essentially, from the mold of Euclidean spaces, finite or infinite-dimensional!) and familiarize themselves with the theory of topological algebras (especially non-normed ones), which may be of great import in many physical applications (Naimark, 1972; Kolmogorov and Fomin, 1975).

Now, in the next paragraph we vindicate some of the remarks above in the light of (Mallios, 2004). In particular, based on Mallios’ ‘universal’ *K*-theoretic musings on 2nd-quantization under the prism of ADG as exposed in that paper, we draw some telling links with our 3rd-quantization of VEG and FYM theories. In addition, we make contact with our (f)initary, ausal and (q)uantal (*:fcq*) VEG developed in the hexalogy (Mallios and Raptis, 2001, 2002, 2003, 2005; Raptis, 2006a,b).

K-theoretic underpinnings of 3rd-quantization and a link with *fcq* ADG-VEG. We can relate the heuristic canonical-type of 3rd-quantization introduced above with Mallios’ *K*-theoretic musings on 2nd-quantization *à la* ADG in Mallios (2004, 2005).

In Mallios (2004), cogent arguments are given for representing the quantum state spaces of (free) elementary particles of quantum (*:2nd-quantized*) fields by the vector sheaves (*:locally free \mathbf{A} -modules of finite rank*) involved in ADG. The syllogism supporting those arguments takes us progressively from *free* (Hilbert) *\mathbf{A} -modules*, to *finitely generated free \mathbf{A} -modules*, to *finitely generated projective \mathbf{A} -modules*, and finally, via an extension of the classical Serre–Swan theorem to non-normed topological algebras,⁶⁹ to vector bundles and their associated vector sheaves \mathcal{E} that ADG is all about.⁷⁰

⁶⁸ As mentioned earlier, at the bottom even of Connes’ functional-analytic/operator-theoretic noncommutative geometry lies the manifold M , with its commutative coordinates.

⁶⁹ The classical Serre–Swan theorem bijectively corresponds finitely generated projective $\mathcal{C}^0(X)$ -modules (with X a compact topological manifold, and $\mathcal{C}^0(X)$ the algebra of continuous \mathbb{C} -valued functions on it) to continuous complex vector bundles over X . Moreover there are smooth analogues of that correspondence, in that one can map bijectively $\mathcal{C}^\infty(X)$ -modules (with X now a compact \mathcal{C}^∞ -smooth manifold) to \mathcal{C}^∞ -vector bundles over it. Mallios’ extension of the classical result consists in allowing for more general (than $\mathcal{C}^\infty(X)$) non-normed topological algebras for coefficient algebras.

⁷⁰ The original motivation for this syllogism was Selesnick’s paper (1983). More technical details of the arguments and their associated constructions, of the closely related ADG-version of the spin-statistics

Regarding our brief remarks at the end of the last paragraph about the use of sheaves instead of bundles and of general (:non-normed) topological algebras instead of Banach ones, two points in Mallios (2004) must be highlighted here, namely:

1. That once one arrives by the above syllogism and its related constructions at vector *sheaves* as a model for the state spaces of field-quanta, one forgets altogether about bundles and works exclusively with the (local) sections of the resulting sheaves;
2. That of special interest (and use!) are certain 'nice' non-normed commutative topological algebras \mathbf{A} ,⁷¹ which are seen to be localized sheaf-theoretically over their *Gel'fand spectra* $\mathfrak{M}(\mathbf{A})$ (Mallios, 1993), which in turn are 'topologically indistinguishable' (e.g., *homotopic*) to the base topological space X over which the \mathbf{A} -module sheaves were defined in the first place. This is a nice example of *Gel'fand duality* and it highlights what we emphasized earlier: that if any 'space(time)' is involved at all in our scheme, it is already encoded in \mathbf{A} , while at the same time one works solely in 'sheaf space' \mathcal{E} (i.e., with \mathcal{E} 's sections) with a purely algebraic (:sheaf-theoretic) 'differential geometric mechanism' that is manifestly \mathbf{A} -functorial. Thus one essentially forgets about 'space(time)' altogether.

Keeping in mind points 1 and 2 above, in Mallios (2004) an elegant K -theoretic formulation and of the Serre–Swan theorem (extended to non-normable topological algebras) is given involving *Grothendieck K -groups*.⁷² In a nutshell, modulo an group isomorphism, one establishes the following equalities

$$K(X) = K(\mathbf{A}) = K(\mathcal{P}(\mathbf{A})) \tag{16}$$

so that X is *homotopic to* $\mathfrak{M}(\mathbf{A})$, while the latter is assumed to be a unital, commutative, associative, locally m -convex Q -algebra (:Waelbroeck algebra).

What is of interest to us here *vis-à-vis* 3rd-quantization is that in Mallios (2004) (16) is further expressed in terms of projection operators. Briefly, one singles out a primitive idempotent linear endomorphism \mathfrak{P}

$$\mathfrak{P} \in M_n(\mathbf{A}) : \mathfrak{P}^2 = \mathfrak{P} \tag{17}$$

connection based on what is there coined *Selesnick's Correspondence*, as well as the various relevant references backing those arguments can be found in Mallios (2004, 2005).

⁷¹ Locally m -convex Q -algebras (*alias*, Waelbroeck algebras) Mallios (1986). An archetypical example of a Waelbroeck algebra is $C^\infty(X)$, with X a compact Hausdorff C^∞ -manifold. Note that here we abuse notation and use also \mathbf{A} for the said algebras (when Mallios uses \mathbb{A}). We hope that the reader will not confuse the 'algebras' with the 'structure sheaves' thereof, but anyway the distinction is going to be clear from the context.

⁷² See Mallios (2004) for details of the constructions, and Mallios (1986) for the technical terminology and notation used therein.

whose kernel $\ker(\mathfrak{P})$ corresponds to a *finitely generated projective \mathbf{A} -module \mathcal{M}*

$$\mathcal{M} = \ker(\mathfrak{P}) \tag{18}$$

defining a Grothendieck class $[\mathcal{M}]$ (of finitely generated projective \mathbf{A} -modules) in $K(\mathbf{A})$. In turn, \mathfrak{P} is seen to lift to a *morphism*

$$\hat{\mathfrak{P}} : X \times \mathbf{A}^n \rightarrow X \times \mathbf{A}^n \tag{19}$$

whose kernel, $[\mathcal{M}] = [\ker(\hat{\mathfrak{P}})]$, is now a Grothendieck class of modules in $K(X)$. Moreover, one easily observes that $\ker(\mathfrak{P})$ defines a continuous finite-dimensional \mathbb{C} -vector bundle (E, π, X) over X , from (the sections of) which one arrives straightforwardly to a vector sheaf \mathcal{E} as defined in ADG.

Regarding our 3rd-quantization scenario of ADG-fields (i.e., vector sheaves carrying connections), the alert reader may have already spotted the ‘caveat’ in the K -theoretic construction above:

The projection operator \mathfrak{P} in (17) may be identified with our sheaf cohomological quantum uncertainty operator \mathfrak{U} in (13).

The full physical meaning and implications of this identification ought to be further explored and better comprehended (Raptis, 2006d).

Finally, since the present paper is a contribution to Sorkin’s 60th birthday fest-volume, it is fitting to make some comments on Mallios’ generalization of the preceding K -theoretic musings (:expressions (5.12) and (5.13) in Mallios (2004)) in the light of our applications of ADG to ‘finitary, causal and quantal’ VEG. In the said expressions, Mallios establishes that

$$K(X) = K(\mathbf{A}) \tag{20}$$

when X is the projective limit of an inverse system $\overleftarrow{\{X_i\}}$ of topological spaces (: $X = \lim_{\leftarrow} X_i$), each of which is the Gel’fand spectrum of a Waelbroeck algebra (: $X_i = \mathfrak{M}(\mathbf{A}_i)$), with the latter constituting an inductive system $\overleftarrow{\{\mathbf{A}_i\}}$ whose direct limit is \mathbf{A} itself (: $\mathbf{A} = \varinjlim \mathbf{A}_i$).

The reader who is familiar with our ‘finitary’ work (Raptis and Zapatrin, 2000, 2001; Raptis, 2000a,b; Mallios and Raptis, 2001, 2002, 2003, 2005; Raptis, 2006a,b), can easily translate the K -theoretic result above to the finitary case where $\overleftarrow{\{X_i\}}$ is taken to be the projective system of Sorkin’s finitary poset substitutes P_i of (a relatively compact region X of) a topological manifold M Sorkin (1991) and $\overleftarrow{\{\mathbf{A}_i\}}$ an inductive system of incidence (Rota) algebras Ω_i , whose primitive Gel’fand spectra are precisely the P_i s ($\mathfrak{M}(\Omega_i) = P_i$).⁷³ This (K -)functorial correspondence has been anticipated in Zapatrin (2001 a,b), and the direct/inverse limits

⁷³ Discrete Gel’fand duality between finitary posets and their incidence algebras (Zapatrin, 1998; Raptis and Zapatrin, 2000, 2001; Raptis, 2000a).

engaged in it have been interpreted as ‘classical limits’ (Raptis and Zapatin, 2000, 2001). These observations too must await a more thorough investigation (Raptis, 2006d).

A final note on terminology. In closing this paper we would like to mention parenthetically that the name ‘*third quantization*’ has already been used in the theoretical physics nomenclature, pertaining to some ideas in early universe cosmology (Strominger, 1991). We should emphasize that our 3rd-quantization has little in common with that original term, so that the two should not be confused or thought to be somehow related. Also, the same term has been used by John Baez in his general theoretical scheme that might be coined ‘higher-order categorical quantization’, *alias*, ‘*n*-th quantization’ ($n \geq 2$) for short (Baez, 1995). Here again, apart from the common formal algebraico-categorical language and technology underlying both our 3rd-quantization and Baez’s, there is little common semantic grounds between the two schemes.

4. SUMMARY WITH CONCLUDING REMARKS

In the present paper we gathered certain central results from manifold applications of ADG to gravity and gauge theories and argued that we already possess a geometrically (pre)quantized, second quantized and manifestly background spacetime manifold independent vacuum Einstein gravitational and free Yang–Mills field dynamics. Based on the *ur* ADG-conception of a field as a pair $(\mathcal{E}, \mathcal{D})$, we entertained the idea of a field quantization scenario called third quantization. 3rd-quantization, like geometric (pre)quantization and second quantization, was seen to be an expressly functorial scheme which, in contradistinction to its two predecessors, does not depend at all on a background manifold for its differential geometric formulation and physical (:spacetime continuum) interpretation. It thus extends them both, following the extension and generalization of the Classical Differential Geometry (CDG) of \mathcal{C}^∞ -smooth manifolds that ADG has achieved by purely algebraico-categorical (:sheaf-theoretic) and sheaf cohomological means.

In what formally looked like canonical quantization, but in the manifest absence of a smooth background geometrical spacetime manifold as befits ADG, we posited abstract non-trivial local Heisenberg-like commutation relations between certain characteristic local (:differential) forms that uniquely characterize sheaf cohomologically the ADG-fields and their particle-quanta. These forms were then physically interpreted in a heuristic way as abstract position and momentum ‘determinations’ (:‘observables’) in accordance with ADG’s (pre)quantum field semantics. The ADG-fields were thus said to be ‘third quantized’, and so are the vacuum Einstein and free Yang–Mills equations that they define differential geometrically as differential equations proper, without any need arising to

quantize an (external to the ADG-fields) spacetime continuum, simply because such a theoretical artifact does not exist in our theory. Due to the explicit functoriality of our ADG-constructions, as well as the background spacetime manifoldness that goes hand in hand with it, 3rd-quantization was seen to be fully covariant and it totally bypasses second quantized gravity's vital reliance on a base M and its diffeomorphism structure group $\text{Diff}(M)$ for its differential geometric formulation and physical interpretation as an external spacetime continuum constrained quantum gauge theory. All in all, we maintained that

ADG-VEG is a purely gauge, external spacetime manifold unconstrained, third quantized theory.

Third quantized ADG-gravity's full covariance and background manifoldness motivates us to view certain outstanding and obstinately resisting (re)resolution current QG problems under a new light. Thus, as a future project we entertain the possibility of developing a genuinely covariant functional integral quantization of vacuum Einstein gravity (and free Yang-Mills theories). The functional integral will be over the moduli space of $\mathcal{Aut}\mathcal{E}$ -equivalent \mathbf{A} -connections, which is the physical configuration space in ADG-gravity. A generalized Radon-type of measure, as recently developed in Mallios (2005), will be used to implement the functional integral. What is more important, however, is to note that, due to the manifest background spacetime manifoldness of ADG-gravity, we expect such an abstract path integral scenario to be free from the problem of finding $\text{Diff}(M)$ -invariant measures on the moduli space of gauge equivalent connections, which has so far stymied the usual manifold and CDG-based path integral approaches. In this way, we will catch glimpses of a genuine equivalence between the 'local' (:differential) canonical-type 3rd-quantized gravity and a potential 'global' functional integral-type of one. Of course, the methods of sheaf theory, especially as they have been developed and used by ADG, enable us to address both local (:differential canonical) and global (path integral) quantization issues. In any case, such an equivalence is formally absent from the usual base spacetime manifold and CDG-based quantization approaches, since, for instance, the smooth base spacetime dependent canonical quantum gravity manifestly breaks covariance in two ways. On the one hand, it mandates a $3 + 1$ space-time split (:foliation of spacetime into spacelike hypersurfaces) in order to concord with a well posed Cauchy problem in the classical theory (:GR) that it purports to quantize, while on the other, in order to adapt consistently the usual canonical formalism and its physical interpretation to the said foliation, it posits equal-time commutation relations between the conjugate gravitational field variables restricted on the aforesaid spatial hypersurfaces. Even in the usual supposedly covariant (:path integral) quantization schemes, whether Lorentzian or Euclidean (which apparently does not formally distinguish between space and time 'directions' *ab initio*), input and output field amplitude

data still have to be specified on initial and final hypersurfaces respectively in order to have a meaningful path integral quantum dynamical propagator between them.

Finally, it is plain that the manifest absence of a background spacetime manifold in 3rd-quantized ADG-VEG prompts us to emphasize that our scheme evades totally the *problem of time* (Isham, 1993; Kuchař, 1993), the *inner product problem*, as well as the *problem of defining meaningful gravitational observables in VEG* (Torre, 1993, 1994), all of which, in one way or another, hinge on our regarding $\text{Diff}(M)$ automorphism ‘structure’ group of the underlying C^∞ -smooth manifold of GR as gauge-constraining the gravitational field, by implementing the PGC.

Declaration. The intuitive and heuristic ideas presented here are ‘raw’ and under development, hence they must await a more formal treatment and a mathematically rigorous exposition (Raptis, 2006d).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge numerous inspiring interactions during a long collaboration with Tasos Mallios on potential applications of ADG to quantum gravity and gauge theories. This work has been financially supported by the European Commission in the form of a generous Marie Curie European Reintegration Grant (ERG-CT-505432) held at the Algebra and Geometry Section, Department of Mathematics, University of Athens (Greece). Material support from the Theoretical Physics Group at Imperial College (London) is also greatly appreciated. Finally, I wish to thank Fay Dowker for giving me the opportunity to contribute a paper to this ‘*Sorkin Fest*’ volume.

REFERENCES

- Ashtekar, A. (1986). New variables for classical and quantum gravity. *Physics Review Letters* **57**, 2244.
- Ashtekar, A. (2003). Quantum geometry and its ramifications. In *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking’s 60th Birthday*, Gibbons, G. W., Shellard, E. P. S., and Rankin, S. J., (eds.), Cambridge University Press, Cambridge.
- Ashtekar, A. and Lewandowski, J. (1995a). Differential geometry on the space of connections via graphs and projective limits. *Journal of Geometry and Physics* **17**, 191.
- Ashtekar, A. and Lewandowski, J. (1995b). Projective techniques and functional integration. *Journal of Mathematical Physics* **36**, 2170.
- Ashtekar, A. and Lewandowski, J. (1997a). Quantum theory of geometry I: Area operators. *Classical and Quantum Gravity* **14**, A55.
- Ashtekar, A. and Lewandowski, J. (1997b). Quantum theory of geometry II: Volume operators. *Advance Theoretical Mathematics Physics* **1**, 388.

- Ashtekar, A. and Lewandowski, (2004). *Background Independent Quantum Gravity: A Status Report*, pre-print; gr-qc/0404018.
- Auyang, S. Y. (1995). *How is Quantum Field Theory Possible?* Oxford University Press, New York-Oxford.
- Baez, J. (1995). *The Story of Nth Quantization*. http://math.ucr.edu/home/baez/nth_quantization.html.
- Baez, J. C. (1994a). Generalized measures in gauge theory. *Letters in Mathematical Physics* **31**, 213.
- Baez, J. C. (1994b). Diffeomorphism invariant generalized measures on the space of connections modulo gauge transformations. In *Proceedings of the Quantum Topology Conference*, Yetter, D., (ed.), World Scientific, Singapore; hep-th/9305045.
- Baez, J. C. (1997). Functional integration on spaces of connections. *Journal of Functional Analysis* **150**, 1.
- Bandyopadhyay, P. (1996). *Geometry, Topology and Quantization*. Kluwer Academic Publishers, Dordrecht-Boston-London.
- Bergmann, P. G. (1979). Unitary field theory: Geometrization of physics or physicalization of geometry? In *The 1979 Berlin Einstein Symposium*, Lecture Notes in Physics, Springer-Verlag, Berlin-Heidelberg-New York .
- Birkhoff, G. D. (1938). Fifty years of american mathematics. *Semicentennial Addresses of the American Mathematical Society*, AMS **307**.
- Block, J. (1998). Review of Alain Connes' book "Noncommutative Geometry" (Connes, 1994). *The Mathematical Intelligencer* **20**(1), 73.
- Bojowald, M. (2001). Absence of singularity in loop quantum cosmology. *Physical Review Letters* **86**, 5227.
- Bombelli, L., Henson, J. and Sorkin, R. D. (2006). *Discreteness Without Symmetry Breaking: A Theorem*, pre-print; gr-qc/0605006.
- Bombelli, L., Lee, J. Meyer, D., and Sorkin, R. D. (1987). Space-time as a causal set. *Physical Review Letters* **59**, 521.
- Bratteli, O. (1979). *Operator Algebras and Quantum Statistical Mechanics, C^* - and W^* -Algebras, Symmetry Groups, Decomposition of States*, **1**. Springer-Verlag, Berlin-Heidelberg-New York.
- Brylinski, J.-L. (1993). *Loop Spaces, Characteristic Classes, and Geometric Quantization*. (Program of Mathematics) **107**, Birkhäuser Boston Inc., Boston, MA.
- Butterfield, J. and Isham, C. J. (2000). Spacetime and the philosophical challenge of quantum gravity. In *Physics Meets Philosophy at the Planck Scale*, Callender, C., and Huggett, N., (eds.), Cambridge University Press, Cambridge; gr-qc/9903072.
- Cao, T. Y. (1997). *Conceptual Developments of 20th Century Field Theories*. Cambridge University Press, Cambridge.
- Chamseddine, A. H. and Connes, A. (1996). Universal formula for noncommutative geometry actions: Unification of gravity and the standard model. *Physical Review Letters* **24**, 4868.
- Chamseddine, A. H. and Connes, A. (1997). The spectral action principle. *Commission Mathematics Physics* **186**, 731.
- Clarke, C. J. S. (1986). Singularities: Local and global aspects. In *Topological Properties and Global Structure of Space-Time*, NATO ASI Series, Bergmann, P. G., and De Sabbata, V., (eds.), Plenum Press, New York and London.
- Clarke, C. J. S. (1993). *The Analysis of Space-Time Singularities*. Cambridge Lecture Notes in Physics, Cambridge University Press, Cambridge.
- Connes, A. (1994). *Noncommutative Geometry*. Academic Press, New York.
- Connes, A. (1998). Noncommutative differential geometry and the structure of spacetime. In *The Geometric Universe* (papers in honour of Roger Penrose), Hugget, S. A., Mason, L. A., Tod, K. P., Tsou, S. T., and Woodhouse, N. M. J., (eds.), Oxford University Press, Oxford.
- Einstein, A. (1952). The foundation of the general theory of relativity. In *The Principle of Relativity*, Dover Publications, New York .

- Einstein, A. (1956). *The Meaning of Relativity*, 5th edn. Princeton University Press, Princeton.
- Echeverria-Enriquez, A., Munoz-Lecanda, M. C., Roman-Roy, N., and Victoria-Monge, C. (1999). *Mathematical Foundations of Geometric Quantization*, pre-print; math-ph/9904008.
- Finkelstein, D. (1969). Space-time code. *Physical Review* **184**, 1261.
- Finkelstein, D. (1988). ‘Superconducting’ causal nets. *International Journal of Theoretical Physics* **27**, 473.
- Finkelstein, D. (1989). Quantum net dynamics. *International Journal of Theoretical Physics* **28**, 441.
- Finkelstein, D. R. (1996). *Quantum Relativity: A Synthesis of the Ideas of Einstein and Heisenberg*. Springer-Verlag, Berlin-Heidelberg-New York.
- Fulling, S. A. (1989). *Aspects of Quantum Field Theory in Curved Space-time*, **17**. London Mathematical Society Student Texts, Cambridge University Press, Cambridge.
- Geroch, R. (1968). What is a singularity in General Relativity? *Annals of Physics* **48**, 526.
- Göckeler, M. and Schücker, T. (1990). *Differential Geometry, Gauge Theories and Gravity*. Cambridge University Press, Cambridge.
- Goldstein, H. (1950). *Classical Mechanics*. Addison-Wesley Publishers, Reading Massachusetts.
- Haag, R. (1996). *Local Quantum Physics: Fields, Particles, Algebras*, 2nd edn. Springer-Verlag, Berlin-Heidelberg-New York.
- Henson, J. (2006a). *The Causal Set Approach to Quantum Gravity*, this volume; gr-qc/0601121.
- Henson, J. (2006b). *Macroscopic Observables and Lorentz Violation in Discrete Quantum Gravity*, pre-print; gr-qc/0604040.
- Husain, V. and Winkler, O. (2004). *Quantum Resolution of Black Hole Singularities*, pre-print; gr-qc/0410125.
- Ivanenko, D. and Sardanashvily, G. (1983). The gauge treatment of gravity. *Physics Reports* **94**, 1.
- Isham, C. J. (1993). Canonical quantum gravity and the problem of time. In *Integrable Systems, Quantum Groups, and Quantum Field Theories*, Kluwer Academic Publishers, London-Amsterdam; gr-qc/9210011.
- Isham, C. J. (2003). A new approach to quantising space-time: I. Quantising on a general category. *Advances Theoretical Mathematics Physics* **7**, 331; gr-qc/0303060.
- Isham, C. J. (2004a). A new approach to quantising space-time: II. Quantising on a category of sets. *Advances Theoretical Mathematics Physics* **7**, 807; gr-qc/0304077.
- Isham, C. J. (2004b). A new approach to quantising space-time: III. State vectors as functions on arrows. *Advances Theoretical Mathematics Physics* **8**, 797; gr-qc/0306064.
- Isham, C. J. (2005). Quantising on a category. *A Festschrift for James Cushing*, to appear; quant-ph/0401175.
- Isidro, J. M. (2004). *Quantum States from Tangent Vectors*, pre-print; quant-ph/0407159.
- Kastler, D. (1986). Introduction to Alain Connes’ Non-Commutative Differential Geometry. In *Fields and Geometry 1986: Proceedings of the XXIIInd Winter School and Workshop of Theoretical Physics, Karpacz, Poland*, Jadczyk, A., (ed.), World Scientific, Singapore.
- Kolmogorov, A. N. and Fomin, A. (1975). *Introductory Real Analysis*. Dover Publications, New York.
- Kribs, D. W. and Markopoulou, F. (2005). *Geometry from Quantum Particles*, pre-print; gr-qc/0510052.
- Kriele, M. (1999). *Spacetime: Foundations of General Relativity and Differential Geometry*, **LNP m59**. Springer-Verlag, Berlin-Heidelberg-New York.
- Kuchař, K. (1993). Time and interpretations of quantum gravity. In *Proceedings of the Fourth Canadian Conference on General Relativity and Relativistic Astrophysics*, Kunstatter, G., Vincent, D., and Williams, J., (eds.), World Scientific, Singapore.
- Leibniz, G. W. (1992). *Discourse on Metaphysics and the Monadology*, translated by Montgomery, G. R., Great Books in Philosophy Series, Prometheus Books, Amherst, New York.
- MacLane, S. (1986). *Mathematics, Form and Function*. Springer-Verlag, New York.
- MacLane, S. and Moerdijk, I. (1992). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, New York.

- Madore, J. (1999). *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, 257, 2nd edn. LMS Lecture Note Series, Cambridge University Press.
- Mallios, A. (1986). *Topological Algebras. Selected Topics*. North-Holland, Amsterdam.
- Mallios, A. (1993). The de Rham-Kähler complex of the Gel'fand sheaf of a topological algebra. *Journal of Mathematical Analysis and Applications* **175**, 143.
- Mallios, A. (1998a). *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, 1–2, Kluwer Academic Publishers, Dordrecht.⁷⁴
- Mallios, A. (1998b). On an axiomatic treatment of differential geometry via vector sheaves. Applications. *Mathematics of Japan (International Plaza)* **48**, 93. (invited paper).
- Mallios, A. (1999). On an axiomatic approach to geometric prequantization: A classification scheme à la Kostant–Souriau–Kirillov. *Journal of Mathematics and Sciences (NY)* **95**, 2648. (invited paper).
- Mallios, A. (2001). Abstract differential geometry, general relativity and singularities. In *Unsolved Problems in Mathematics for the 21st Century: A Tribute to Kiyoshi Iséki's 80th Birthday*, 77, Abe, J. M., and Tanaka, S., (eds.), IOS Press, Amsterdam. (invited paper).
- Mallios, A. (2005). *Modern Differential Geometry in Gauge Theories. Vol. I: Maxwell Fields, Vol. II: Yang–Mills Fields*, 2-volume continuation (including abstract integration theory) of Mallios (1998a), Birkhäuser, Boston-Basel-Berlin.
- Mallios, A. (2004). K-Theory of topological algebras and second quantization. *Acta Universitatis Ouluensis—Scientiae Rerum Naturalium* **A408**, 145; math-ph/0207035.
- Mallios, A. (2002). Abstract differential geometry, singularities and physical applications. In *Topological Algebras with Applications to Differential Geometry and Mathematical Physics*, in Proceedings of a Fest-Colloquium in Honour of Professor Anastasios Mallios (16–18/9/1999), Strantzalos, P., and Fragouloupoulou, M., (eds.), Department of Mathematics, University of Athens Publications.
- Mallios, A. (2006a). Quantum gravity and “singularities”. Note *Matematica*, **25**, 57 (2006) (invited paper); physics/0405111.
- Mallios, A. (2006b). *Geometry and Physics Today*, to appear in a Special Proceedings issue for *Glaifka–2004: Iconoclastic Approaches to Quantum Gravity*, Raptis, I., (ed.), International Journal of Theoretical Physics (invited paper); gr-qc/0506088.
- Mallios, A. (2002). *Remarks on “singularities”*, (pre-print); gr-qc/0202028.
- Mallios, A. and Raptis, I. (2001). Finitary spacetime sheaves of quantum causal sets: Curving quantum causality. *International Journal of Theoretical Physics* **40**, 1885; gr-qc/0102097.
- Mallios, A. and Raptis, I. (2002). Finitary Čech-de Rham cohomology. *International Journal of Theoretical Physics* **41**, 1857; gr-qc/0110033.
- Mallios, A. and Raptis, I. (2003). Finitary, causal and quantal vacuum Einstein gravity. *International Journal of Theoretical Physics* **42**, 1479; gr-qc/0209048.
- Mallios, A. and Raptis, I. (2005). *C^∞ -Smooth Singularities Exposed: Chimeras of the Differential Spacetime Manifold*, research monograph (in preparation); gr-qc/0411121.⁷⁵
- Mallios, A. and Rosinger, E. E. (1999). Abstract differential geometry, differential algebras of generalized functions and de Rham cohomology. *Acta Applied Mathematics* **55**, 231.
- Mallios, A. and Rosinger, E. E. (2001). Space-time foam dense singularities and de Rham cohomology. *Acta Applied Mathematics* **67**, 59.
- Mallios, A. and Rosinger, E. E. (2002). Dense singularities and de Rham cohomology. In *Topological Algebras with Applications to Differential Geometry and Mathematical Physics*, in Proceedings

⁷⁴ There is also a Russian translation of this 2-volume book by MIR Publishers, Moscow (vol. 1, 2000 and vol. 2, 2001).

⁷⁵ Preliminary, last year's version posted at gr-qc.

- of a Fest-Colloquium in Honour of Professor Anastasios Mallios (16–18/9/1999), Strantzalos, P., and Fragoulopoulou, M., (eds.), Department of Mathematics, University of Athens Publications.
- Manin, Y. I. (1988). *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin-Heidelberg-New York.
- Modesto, L. (2004). *Disappearance of the Black Hole Singularity in Quantum Gravity*, pre-print; gr-qc/0407097.
- Naimark, A. M. (1972). *Normed Algebras*. Wolters-Noordhoff, Groningen.
- Raptis, I. (2000a). Algebraic quantization of causal sets. *International Journal of Theoretical Physics* **39**, 1233; gr-qc/9906103.
- Raptis, I. (2000b). Finitary spacetime sheaves. *International Journal of Theoretical Physics* **39**, 1703; gr-qc/0102108.
- Raptis, I. (2006a). Finitary-Algebraic ‘Resolution’ of the inner schwarzschild singularity. *International Journal of Theoretical Physics* **45**(6), (to appear); gr-qc/0408045.
- Raptis, I. (2006b). Finitary topos for locally finite, causal and quantal vacuum einstein gravity. *International Journal of Theoretical Physics* (submitted); gr-qc/0507100.
- Raptis, I. (2006c). *Categorical Quantum Gravity*, to appear in a Special Proceedings issue for *Glařka–2004: Iconoclastic Approaches to Quantum Gravity*, Raptis, I., (ed.), *International Journal of Theoretical Physics*; gr-qc/0509089.
- Raptis, I. (2006d). *Sheaf Cohomological Quantization of Gravity and Gauge Theories Based on Abstract Differential Geometry* (work in progress).
- Raptis, I. and Zapatrin, R. R. (2000). Quantization of discretized spacetimes and the correspondence principle. *International Journal of Theoretical Physics* **39**, 1; gr-qc/9904079.
- Raptis, I. and Zapatrin, R. R. (2001). *Algebraic description of spacetime foam. Classical and Quantum Gravity* **20**, 4187; gr-qc/0102048.
- Rovelli, C. and Smolin, L. (1990). Loop space representation of quantum general relativity. *Nuclear Physics* **B331**, 80.
- Rovelli, C. and Smolin, L. (1995). Discreteness of area and volume in quantum gravity. *Nuclear Physics* **B442**, 593. [Erratum: *Nuclear Physics*, **B456**, 734 (1995).]
- Russell, B. (1948). *Human Knowledge: Its Scope and Limits*, Allen & Unwin, London.
- Seiberg, N. (2006). *Emergent Spacetime*, pre-print; hep-th/0601234.
- Selesnick, S. A. (1983). Second quantization, projective modules, and local gauge invariance. *International Journal of Theoretical Physics* **22**, 29.
- Smolin, L. (2004) *An Invitation to Loop Quantum Gravity*, pre-print; gr-qc/0408048.
- Smolin, L. (2005). *The Case for Background Independence*, pre-print; hep-th/0507235.
- Sorkin, R. D. (1991) Finitary substitute for continuous topology. *International Journal of Theoretical Physics* **30**, 923.
- Sorkin, R. D. (1995). A specimen of theory construction from quantum gravity. In *The Creation of Ideas in Physics*, Leplin, J., (ed.), Kluwer Academic Publishers, Dordrecht; gr-qc/9511063.
- Sorkin, R. D. (1997). Forks in the road, on the way to quantum gravity. *International Journal of Theoretical Physics* **36**, 2759; gr-qc/9706002.
- Sorkin, R. D. (2003). Idecomposable ideals in incidence algebras. *Modern Physics Letters* **A18**, 2491; math.CO/0309126.
- Stachel, J. J. (1987). How Einstein discovered general relativit: A historical tale with some contemporary morals. In *Proceedings of the 11th International Conference on General Relativity and Gravitation*, MacCallum, M. A. H., (ed.), Cambridge University Press, Cambridge.
- Stachel, J. J. (2003). A brief history of space-time. In *2001: A Relativistic Spacetime Odyssey*, Ciuffolini, I., Dominici, D., and Lusanna, L., (eds.), World Scientific, Singapore (2003).⁷⁶

⁷⁶This paper was e-communicated in pre-print form to me by Rafael Sorkin. I am greatly indebted to him for this communication.

- Strominger, A. (1991). Baby universes. In *Quantum Cosmology and Baby Universes, Proceedings of the Jerusalem Winter School for Theoretical Physics*, Coleman, S., Hartle, J. B., Piran, T., and Weinberg, S., (eds.) World Scientific, Singapore-London-Hong Kong.
- Thiemann, T. (1998). A length operator for canonical quantum gravity. *Journal of Mathematical Physics* **39**, 3372.
- Thiemann, T. (2001). *Introduction to Modern Canonical Quantum General Relativity*, pre-print; gr-qc/0110034.
- Thiemann, T. (2002). *Lectures on Loop Quantum Gravity*, pre-print; gr-qc/0210094.
- Torre, C. G. (1993). Gravitational observables and local symmetries. *Physical Review* **D48**, 2373.
- Torre, C. G. (1994). *The Problem of Time and Observables: Some Recent Mathematical Results*, pre-print; gr-qc/9404029.
- Vassiliou, E. (1994). On Mallios' \mathcal{A} -connections as connections on principal sheaves. *Note Matematica* **14**, 237.
- Vassiliou, E. (1999). Connections on principal sheaves. In *New Developments in Differential Geometry*, Szenthe, J., (ed.) Kluwer Academic Publishers, Dordrecht.
- Vassiliou, E. (2000). On the geometry of associated sheaves. *Bulletin Greek Mathematical Society* **44**, 157.
- Vassiliou, E. (2005). *Geometry of Principal Sheaves*, Springer-Verlag, Berlin-Heidelberg-New York.
- Woodhouse, N. M. J. (1997). *Geometric Quantization*, 2nd edn. Clarendon Press, Oxford.
- Zapatrin, R. R. (1998). Finitary algebraic superspace. *International Journal of Theoretical Physics* **37**, 799.
- Zapatrin, R. R. (2001a). Incidence algebras of simplicial complexes. *Pure Mathematics and Applications* **11**, 105; math.CO/0001065.
- Zapatrin, R. R. (2001b). *Continuous limits of discrete differential manifolds*, pre-print.⁷⁷
- Zeh, H. D. (2003). There is no "first" quantization. *Physics Letters* **A309**, 329; quant-ph/0210098.

POSTSCRIPT: BRIEF ENCOUNTERS OF THE THIRD KIND

It is with great pleasure that I contribute the present paper in honour of Rafael Sorkin. In what follows, I would like to sketch out perigrammatically a short memoir of my crossing worldlines, interaction vertices and scattering cross-sections with Rafael and his work. In keeping with the central notions of gauge theory of the 3rd kind and 3rd-quantization presented in this paper, the personal account of my brief 'other worldly' experiences with Ray and his research may be fittingly coined '*brief encounters of the 3rd kind*'.

Meeting worldlines and interaction vertices.. My interest in Rafael's work began quite unexpectedly back in 1992 when, as a first-year doctoral student at the University of Newcastle upon Tyne, I accidentally bumped into a pre-print of a Greek sounding physicist—Charilaos Aneziris—titled *Topology and Statistics in Zero Dimensions*. In that paper, some interesting links between discrete topology and the quantum spin-statistics connection were drawn. The work was based primarily on a paper by Rafael Sorkin titled '*Finitary Substitute for Continuous Topology*' (abbr. FSCT hereafter) written a year or so earlier (Sorkin, 1991).

⁷⁷This pre-print can be retrieved from Roman Zapatrin's personal webpage, at: www.isiosf.isi.it/~zapatrin.

I thus tracked down the latter via British Interlibrary Loans (for the small Armstrong library at Newcastle did not keep an up-to-date stock of IJTP) and found it profound and masterfully written. I was particularly impressed by the simplicity of ideas and the 'unassuming' character of Rafael's writing. I was (I guess I still am) a novice in QG research, thinking that the Holy Grail of modern theoretical physics would somehow have to involve intricate physical reasoning, dressed up in a fancy, almost cryptic, mathematical language. I was dumbfounded to find Rafael's 'finitary stuff' deep, yet simple; original and fresh, yet as if I had subconsciously already thought about it somehow (or at least, I felt ready to sit down and do research on it!).⁷⁸

In the FSCT, I first came across the causal set (:causet) scenario. I then read the seminal '*Bomb Lee, Me and Sorkin*' paper (Bombelli *et al.*, 1987) and for a while I got hooked on causets, if only day-dreaming about them. At about the same time, I came across David Finkelstein's work on the '*Space-Time Code*' (Finkelstein, 1969) and his '*Superconducting Causal Nets*' (Finkelstein, 1988) (the second written two decades after the first), in which I found the primitive seeds for a '*quantum algebraization of discrete causality*'. Then, I recall my first brief *têt-a-têt* meeting with Rafael at a coffee break during the 3rd Quantum Concepts in Space and Time conference, organized by Chris Isham and Roger Penrose, in Durham (July 1994). There, I doubt whether Rafael remembers our fleeting encounter, I recall pitching to him the idea of algebraizing discrete causality *à la* Finkelstein, and of the general idea of finding an algebraic structure to encode a locally finite poset (be it a finitary topological substitute of the continuum, or a causet).

Already a decade earlier, however, there was a sea-change in Rafael's thinking about locally finite partial orders: from their original inception as coarse topological approximations of the spacetime continuum, to their being regarded as fundamental discrete causal structures to which, in the other way round, the Lorentzian spacetime manifold of GR is now a coarse approximation. This reversal in the physical interpretation of finitary porders is nicely accounted for in (Sorkin, 1995).⁷⁹ On the other hand, the FSCT paper made a deep and lasting impression on me in that the primitive idea was suggested to replace the operationally ideal and 'singular' points of the point-set spacetime continuum by 'fat' regions (:open sets) about them, the latter belonging to locally finite coverings of the topological manifold we started with. Then, relative to such covers, Rafael extracted a finitary poset, which, moreover, had the structure of a 'pointless' simplicial complex. It is

⁷⁸ Indeed, I later realized that the profundity of Rafael's papers lies in their laconic, 'ostensive', almost '*in-your-face*', style and subject matter. They lay bare what is at stake and they expose their subject in the simplest *conceptual* language possible, virtues that I had previously encountered only in (some, mainly philosophical) post-30s Einstein writings, as well as in (some of) Feynman's Lectures in Physics.

⁷⁹ Another example of a simple, fluent and conceptually deep paper.

not an exaggeration to say that the said ‘pointlessness’ and ‘algebraicity’ were two of my original motivations for applying category and in particular topos-theoretic ideas to QG in my Ph.D. thesis. Such was Rafael’s influence.

Soon after I got my doctorate, I became familiar with Tasos Mallios’ ADG theory, and as a postdoc at the maths department of the University of Athens I met Roman Zapatrin who gave a most interesting seminar on a possible algebraization of Rafael’s finitary-topological posets (:fintoposets) using so-called *incidence (Rota) algebras* (Zapatrin, 1998). Roman invited me to St-Petersburg, where we wrote our first collaborative paper on a possible algebraic quantization scheme for the fintoposets of the FSCT paper based on the incidence algebras thereof (Raptis and Zapatrin, 2000). Shortly after, by analogy to the topological case, but now bearing in mind the aforesaid semantic reversal in (Sorkin, 1995), I conceived of a similar ‘algebraic quantization’ scenario for causets (Raptis, 2000a) using a discrete version of *Gel’fand duality* originally due to Roman.

Here I shall digress a bit and tell you a telling little anecdote: in the early summer of the millennium year (:six years after I had first met Rafael in Durham!), when I was a maths postdoc at the University of Pretoria, I e-mailed Rafael, excited about my algebraic quantization of causets scheme. I never received any reply from him during the whole summer, thus I was quite disappointed and thought that my ideas were not that interesting to causet people after all. However, in mid-September I unexpectedly received the following laconic, almost telegraphic, 2-line e-message:⁸⁰

“ I know of your work. You formulated Gel’fand duality for causets before Djamel Dou⁸¹ and I did, thus there is no need for us to ‘beat around the bush’.

This message highlights nicely the following triptych of traits of Rafael’s character (of course, with a bit of generalization written in inverted commas below):

1. He ‘always’ answers laconically and to the point;
2. He ‘never’ answers to his e-mail messages promptly;
3. He is ever ready to acknowledge the work of other people and to give credit, when credit is due;⁸²
4. In retrospect, especially after the appearance of his fairly recent paper (Sorkin, 1997), it is plain to me that Rafael never regards a robust and beautiful result, such as discrete Gel’fand duality for finitary posets, as ‘closing the matter’ (i.e., that there is no need of ‘beating around the

⁸⁰ The following is a reconstruction from memory of Rafael’s message, but it is pretty accurate (especially the last 4-word expression).

⁸¹ A doctoral student-collaborator of his at Syracuse University back then, I believe.

⁸² Although in this case, proper credit should have been given to Roman, for the Gel’fand duality for causets (Raptis, 2000a) comes *mutatis mutandis* from the corresponding duality between finitary topological posets (:simplicial complexes) and their incidence algebras (Zapatrin, 1998; Raptis and Zapatrin, 2000, 2001; Zapatrin, 2001a).

bush’). For the bush is always out there to be beaten,⁸³ in the sense that a result can always be improved, hence for instance his discovery of ideals in incidence algebras better suited to the causet structure and its physical semantics than our Gel’fand ideals (Sorkin, 2003).

*Scattering cross-sections and diverging amplitudes.*⁸⁴ I noted earlier my coming across Mallios’ ADG in the late 90s. Thereafter, my principal research interests have focused primarily on applying the latter to the finitary topological, as well as the causet, scenaria for Lorentzian QG. Thus, after those initial interactions with Rafael, the resulting scattering saw us taking slightly different paths towards QG. However, no matter what the future brings, no matter how much our (re)search (ad)ventures may seem to differ or diverge from each other, Rafael’s paradigmatic figure as a research scientist and exemplary manner as a human being in general—his calm, low-key demeanor and mild tone of voice; his giving you the feeling that he is listening to you quietly, but attentively and thoughtfully; his impressively deep understanding and broad knowledge of all the different approaches to QG (and there is a wild zoo out there!); his original, uncompromising and ‘iconoclastic’ causet research programme⁸⁵; his kind, friendly, yet intense, almost ascetic, face, as well as his polite and inviting manner—will always be with me to inspire and guide my quests. All in all, I consider myself extremely privileged and fortunate to have met Rafael personally, and to have engaged, even if just for a short while, into deep inelastic scattering with him about QG matters; albeit, well above Planck length(!)

So, *belated happy 60th birthday, Rafael*: may you keep on showing us the way to QG for many years to come, in spite of the numerous ‘forks in the road’ (Sorkin, 1997), or of the Sirens’ song of other currently more fashionable QG research programmes, that may ultimately (mis)lead us astray.

⁸³ Part and parcel, I guess, of Rafael’s ‘hard-core’ physico-philosophical realism.

⁸⁴ I am quite sure that Rafael, who is the epitome of modesty, won’t feel comfortable with the verbose eulogy that follows. I apologize in advance, but in a way I feel ‘obliged’ to do it, plus I don’t know of another way of expressing what I wish to say and what I feel.

⁸⁵ Which, lately, has been growing from strength to strength, both from gathering significant results and from gaining popularity.